Robust Exchange Rates and The International Entropy Frontier

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Motivation

- **Goal**: understand the role of concern for model misspecification in international finance
- Study an economy with:
  - complete markets
  - multiple goods
  - robust preferences
- We find that
  - International Risk Sharing involves variances, skewness, kurtosis,...
  - Endogenous disagreement about distribution of fundamentals
  - Endogenous time-variation in volatility of FX rates

Roadmap

- Setup of the Economy
- The International Mean-Entropy Frontier
- Planner’s problem and relevant state variables
- Distorted probabilities and endogenous disagreement
- Robust Exchange Rates

Setup of the economy

- Each agent $i \in \{h, f\}$ has a preference for robustness

$$U_{i,t} = (1-\delta)\log C_{i,t} + \delta\theta \log E_t \exp\left\{\frac{U_{i,t+1}}{\theta}\right\}$$

where $\theta < 0$ measures the degree of concern about model misspecification.

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where $\theta < 0$ measures the degree of concern about model misspecification. If $\theta \to -\infty$: Expected Utility case.

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$$U_{i,t} \approx (1-\delta)\log C_{i,t} + \delta E_t[U_{i,t+1}] + \frac{\delta V_t[U_{i,t+1}]}{2\theta} + \frac{\delta E_t(U_{i,t+1} - E_t U_{i,t+1})^3}{6\theta^2} \dots$$

where $\theta < 0$ measures the degree of concern about model misspecification. Conditional Moments matter.

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- Preferences are defined over the consumption aggregate

$$C_{h,t} = (x_{h,t})^{\alpha} (y_{h,t})^{1-\alpha} \quad \text{and} \quad C_{f,t} = (x_{f,t})^{1-\alpha} (y_{f,t})^{\alpha}$$

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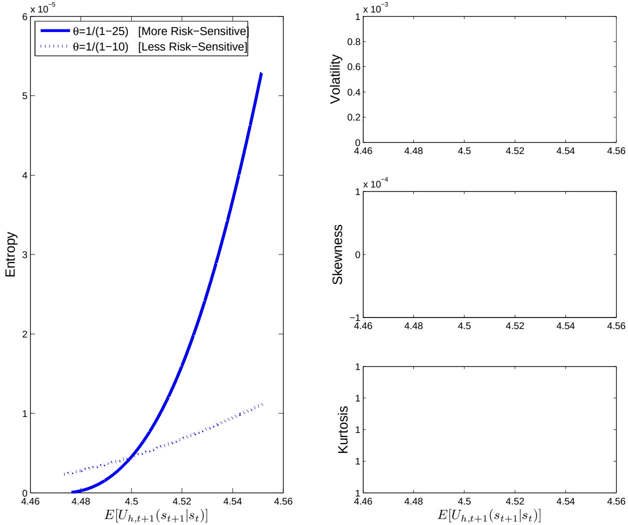
  2. Rare events Details

The International Mean-Entropy Frontier (Two states)



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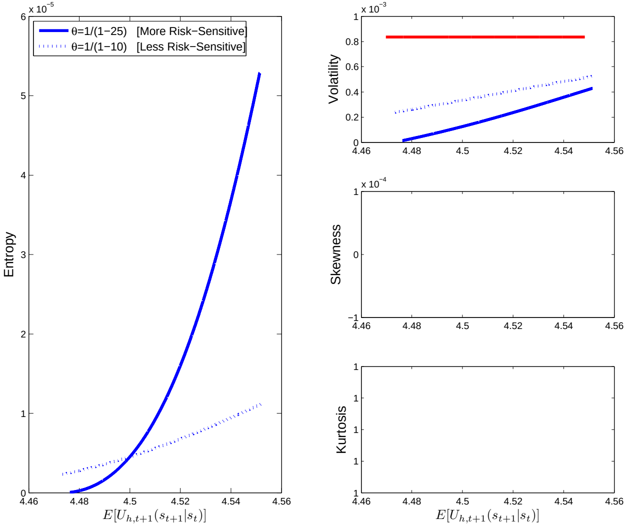

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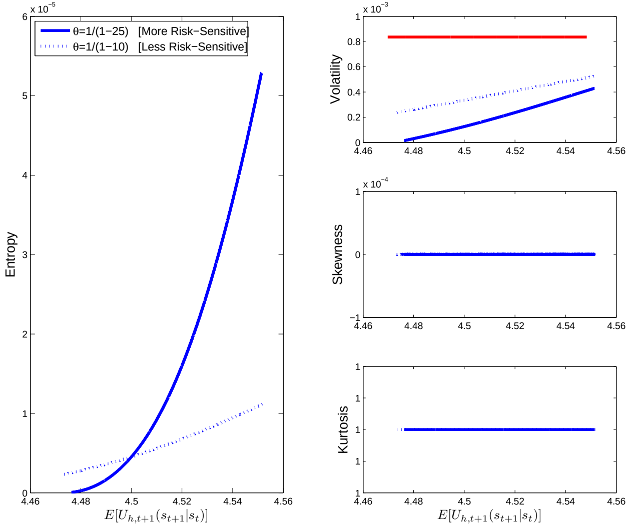
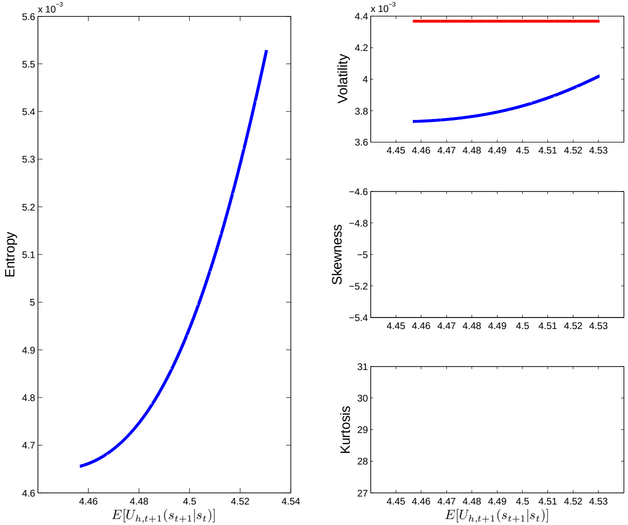


The International Mean-Entropy Frontier (Rare Events)



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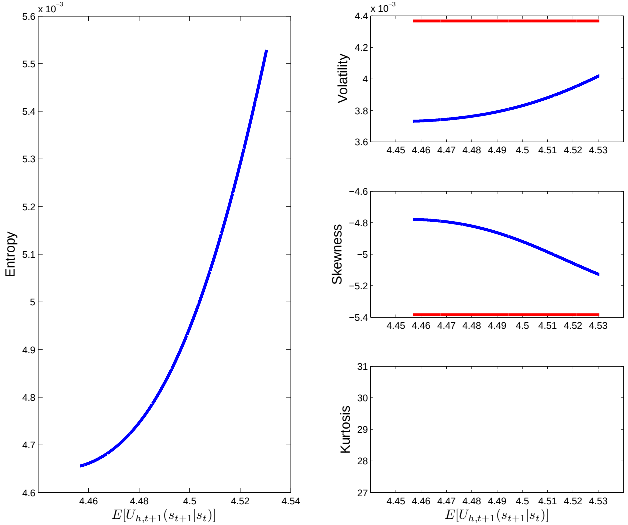

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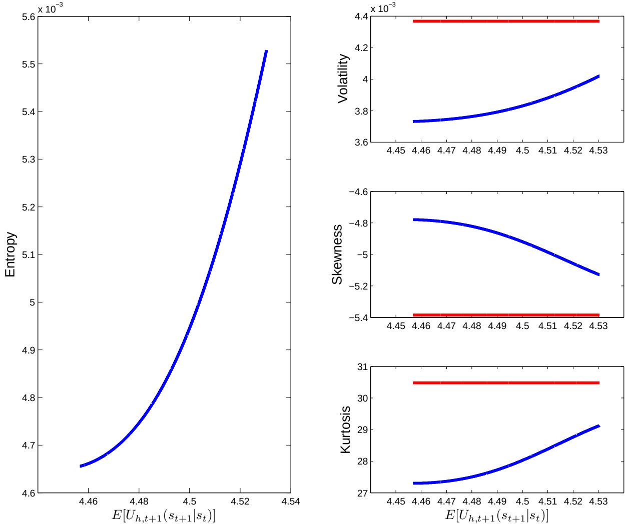

Planner's problem

Efficient allocations are the solution to the planner's problem

$$\begin{aligned}
\text{choose} \quad & \{x_{h,t}, x_{f,t}, y_{h,t}, y_{f,t}\}_{t=0}^{+\infty} \\
\text{to max} \quad & Q = \mu_h U_{h,0} + \mu_f U_{f,0} \\
\text{s.t.} \quad & x_{h,t} + x_{f,t} = X_t \\
& y_{h,t} + y_{f,t} = Y_t, \quad \forall t \geq 0
\end{aligned}$$

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- μ_h and μ_f correspond to an initial distribution of assets.
- Notation: $S = \mu_h / \mu_f$.

Allocations

Time Additive Preferences

Let $k = \frac{\alpha}{1-\alpha}$:

$$x_t^h = \frac{kS}{1+kS} X_t, \qquad x_t^f = \frac{1}{1+kS} X_t$$
$$y_t^h = \frac{S}{k+S} Y_t, \qquad y_t^f = \frac{k}{k+S} Y_t$$

where

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Allocations

Risk Sensitive Preferences

Let $k = \frac{\alpha}{1-\alpha}$:

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$$y_t^h = \frac{S_t}{k+S_t} Y_t, \qquad y_t^f = \frac{k}{k+S_t} Y_t$$

where

$$S_t \quad = \quad S_{t-1} \cdot \frac{\delta \exp\{U_{h,t}/\theta\}}{E_{t-1} \exp\{U_{h,t}/\theta\}} \Bigg/ \frac{\delta \exp\{U_{f,t}/\theta\}}{E_{t-1} \exp\{U_{f,t}/\theta\}}$$

Properties of the Pareto weights

Pareto weights are:

1. countercyclical ▸ Graph
2. expected to increase (decrease) when they are low (high) ▸ Graph
3. stationary ▸ Graph

Distorted Probabilities and Disagreement

- Concern for misspecification of the endowments' distribution translates into distorted probabilities

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- Worst case distortion is **state**-specific

$$\begin{aligned}\widetilde{\pi}^{HL}_{i,t+1} &= \pi^{HL}_{t+1}\frac{\exp\left\{U^{HL}_i(s^{t+1})/\theta\right\}}{\sum_{s_{t+1}}\exp\left\{U_i(s_{t+1}|\mu_{h,t})/\theta\right\}\pi(s_{t+1})} \\ \widetilde{\pi}^{LH}_{i,t+1} &= \pi^{LH}_{t+1}\frac{\exp\left\{U^{LH}_i(s^{t+1})/\theta\right\}}{\sum_{s_{t+1}}\exp\left\{U_i(s_{t+1}|\mu_{h,t})/\theta\right\}\pi(s_{t+1})}, \quad \forall i\in\{h,f\}\end{aligned}$$

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- Worst case distortion is **state**-specific and **country**-specific:

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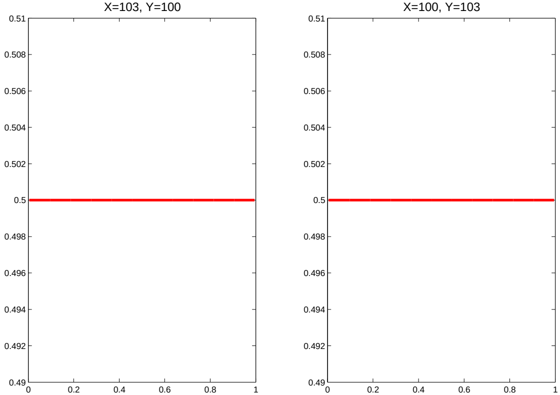
- International Disagreement as an endogenous outcome.

Distorted Probabilities (Two states)

Home Country



Distorted Probabilities (Two states)

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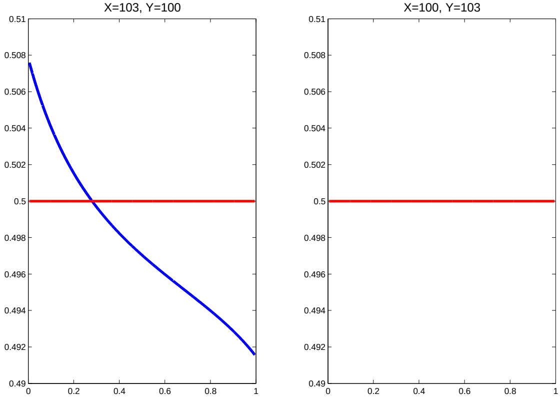

$\longrightarrow$ Distorted probability of high endowment of good X is decreasing;

Distorted Probabilities (Two states)

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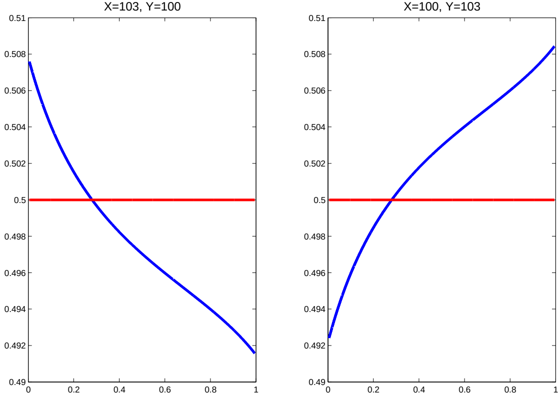

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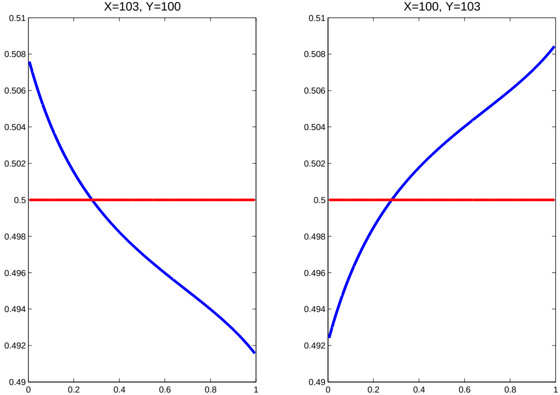


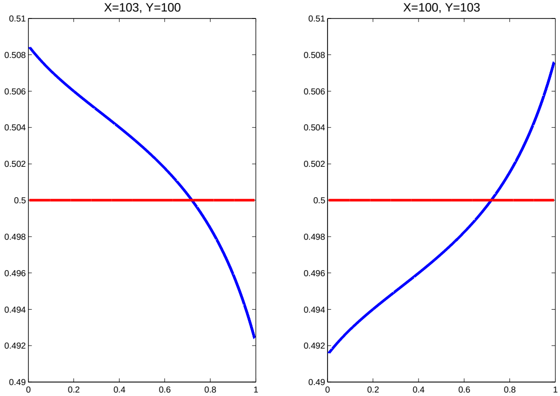
$\longrightarrow$ Distorted probability of high endowment of good X is decreasing;

$\longrightarrow$ Distorted probability of high endowment of good Y is increasing;

$\longrightarrow$ $\widehat{\pi}^{HL} \lesseqgtr \widehat{\pi}^{LH}$ depends on worst case induced by risk-sharing.

Distorted Probabilities (Two states)

Foreign Country



$\longrightarrow$ Foreign country's distorted probabilities are mirror image

Distorted Probabilities (Rare Events)

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Home Country

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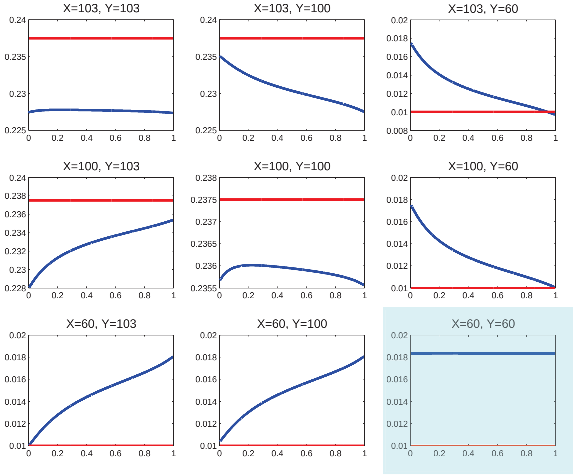
Home Country



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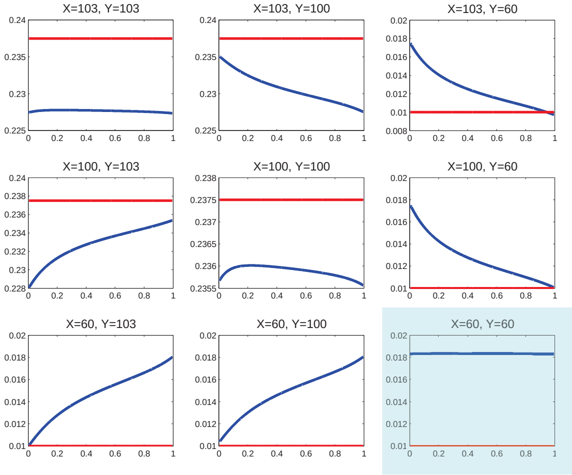
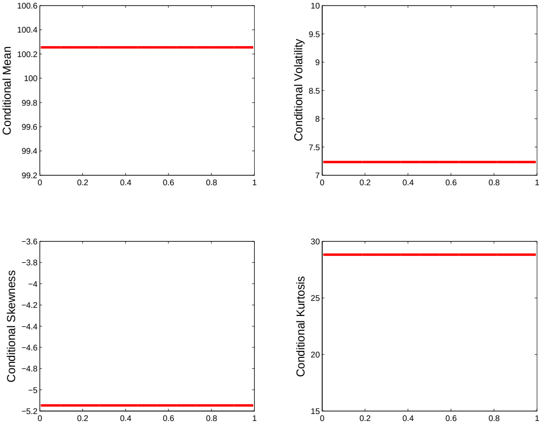
Home Country



$\longrightarrow$ Distorted probability of joint disaster is very large.

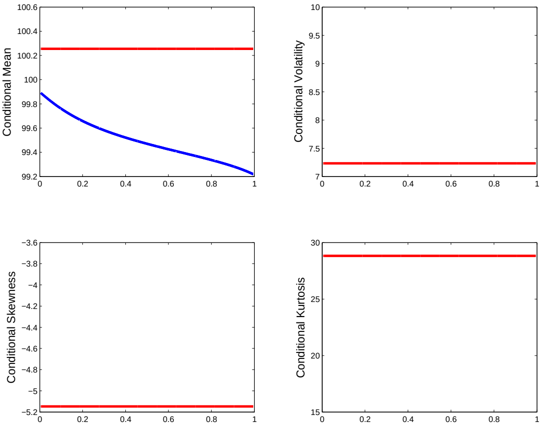
Distorted moments (Rare Events)

Home Country, Home Good (X)



Distorted moments (Rare Events)

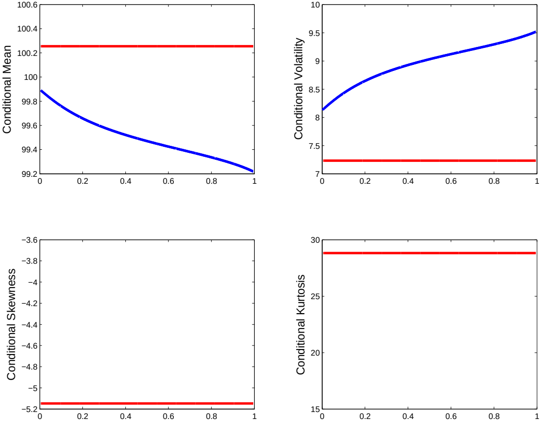
Home Country, Home Good (*X*)



$\longrightarrow$ Conditional Mean is decreasing

Distorted moments (Rare Events)

Home Country, Home Good (X)



$\longrightarrow$ Conditional Volatility is higher than true volatility

Distorted moments (Rare Events)

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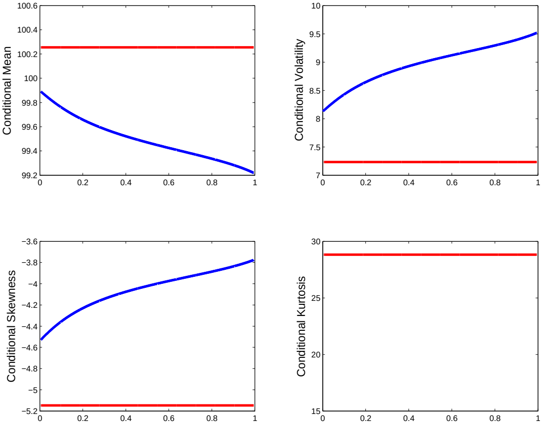
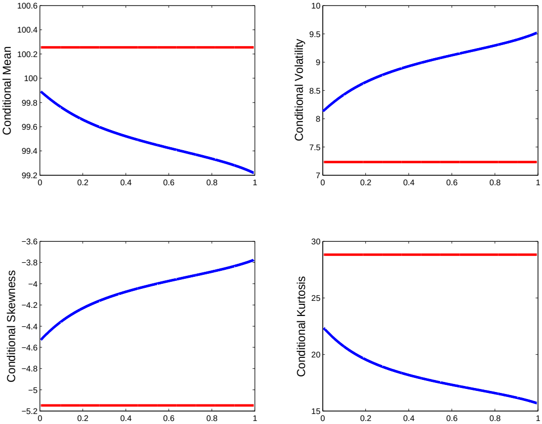


$\longrightarrow$ Conditional Skewness is higher than true one ▸ Why?

Distorted moments (Rare Events)

Home Country, Home Good (X)



$\longrightarrow$ Conditional Skewness is higher than true one ▸ Why?

$\longrightarrow$ Conditional Kurtosis is lower than true one ▸ Why?

Robust FX

$$\Delta e_{t+1} = m_{f,t+1} - m_{h,t+1}$$

Robust FX

$$\boxed{V_t[\Delta e_{t+1}] = V_t[m_{f,t+1} - m_{h,t+1}]}$$

Robust FX

$$V_t[\Delta e_{t+1}] = V_t[m_{f,t+1}] + V_t[m_{h,t+1}] - 2\rho_t \cdot \sqrt{V_t[m_{f,t+1}]} \cdot \sqrt{V_t[m_{h,t+1}]}$$

Robust FX

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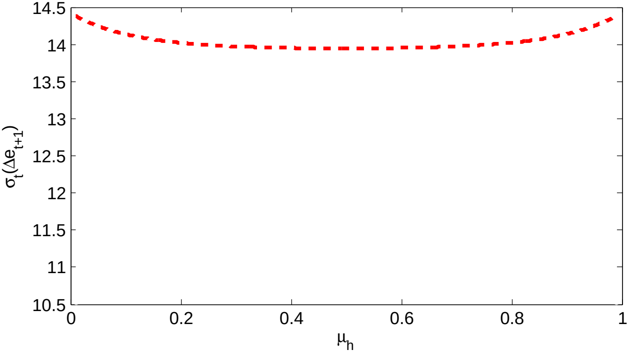

$\longrightarrow$ Average Volatility $\approx$ 14%

Robust FX

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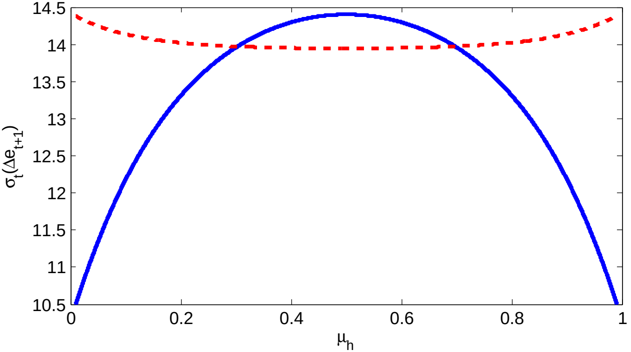
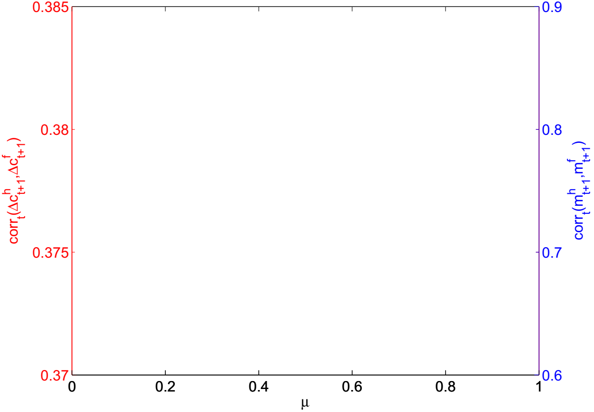


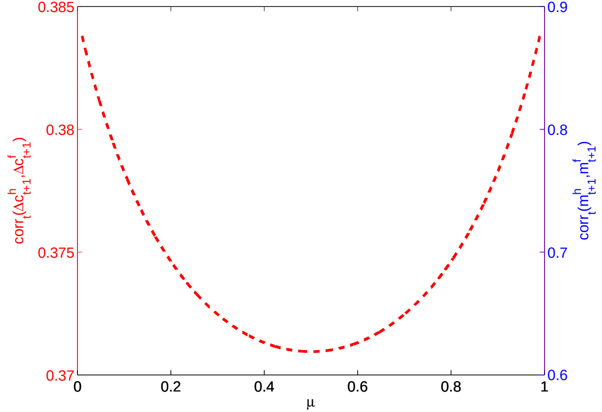
→ Average Volatility ≈ 14%

→ Time-varying exchange rate volatility

Conditional Correlations

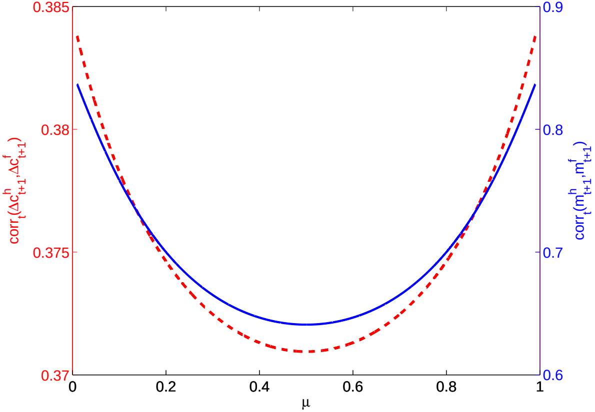


Conditional Correlations



$\longrightarrow$ Low, time-varying correlation of consumption

Conditional Correlations



→ Low, time-varying correlation of consumption

→ High, time-varying correlation of marginal utilities

Concluding Remarks

- Robust International Risk Sharing generates
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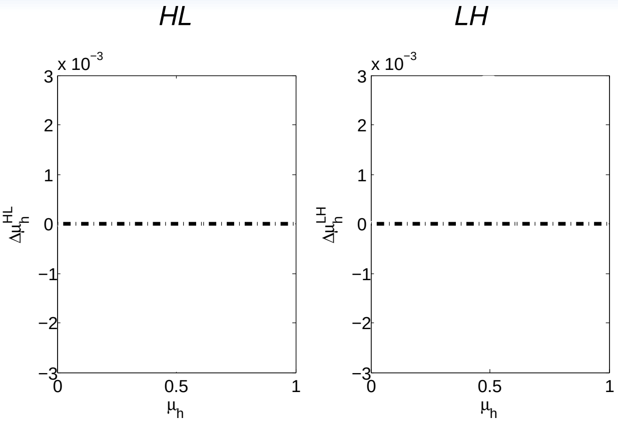
  - more than 2 countries?
  - heteroskedastic endowments?

Pareto weights: phase diagrams

Back



Pareto weights: phase diagrams

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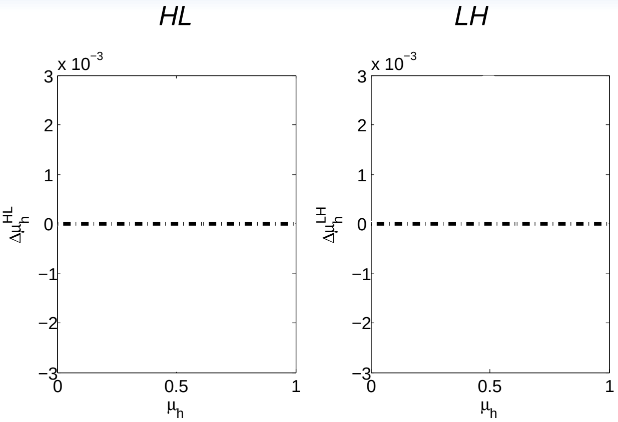

→ **Abundant** X, scarce Y:

Pareto weights: phase diagrams

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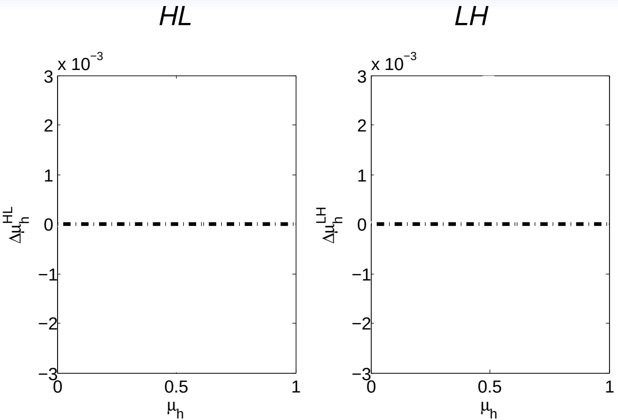

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→ Good news for home

Pareto weights: phase diagrams

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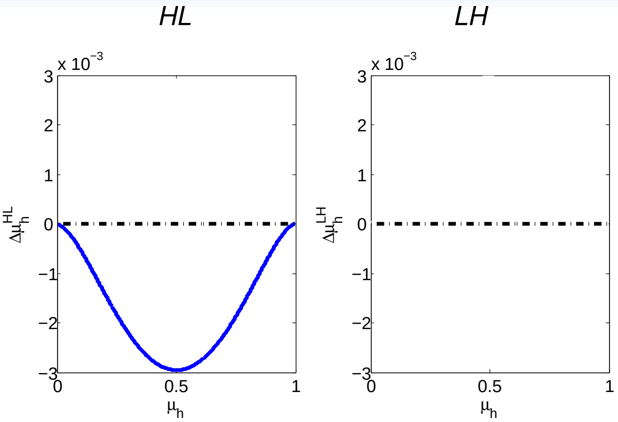

→ **Abundant** *X*, scarce *Y*:

→ Good news for home

→ Home Pareto weight ↓

Pareto weights: phase diagrams

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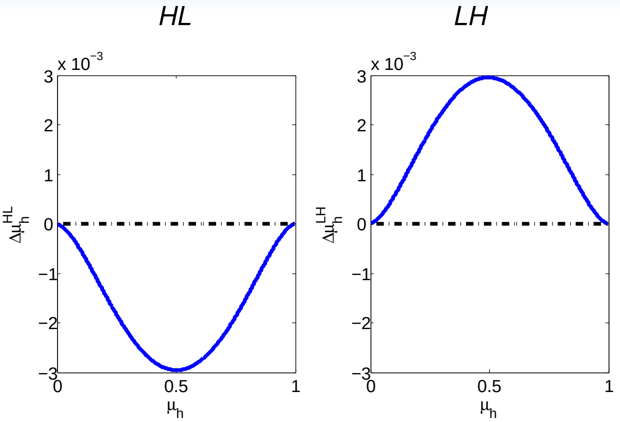


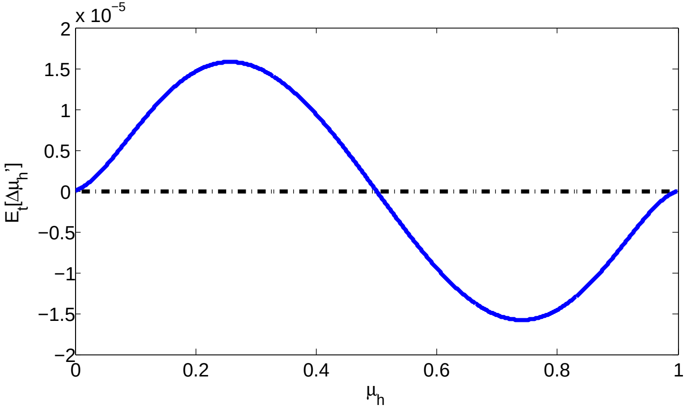
→ **Scarce** *X*, abundant *Y*:

→ Bad news for home

→ Home Pareto weight ↑

Pareto weights: expected change

Back



Pareto weights: expected change

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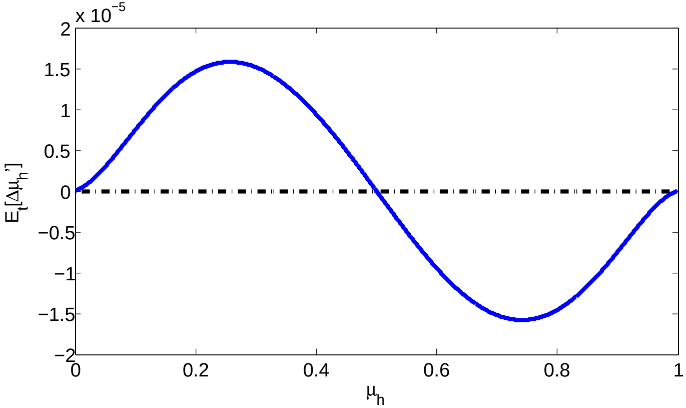

$\rightarrow$ $E_t\left[\mu_{h,t+1}\right] > \mu_{h,t}$, if $\mu_{h,t} \leq 1/2$

Pareto weights: expected change

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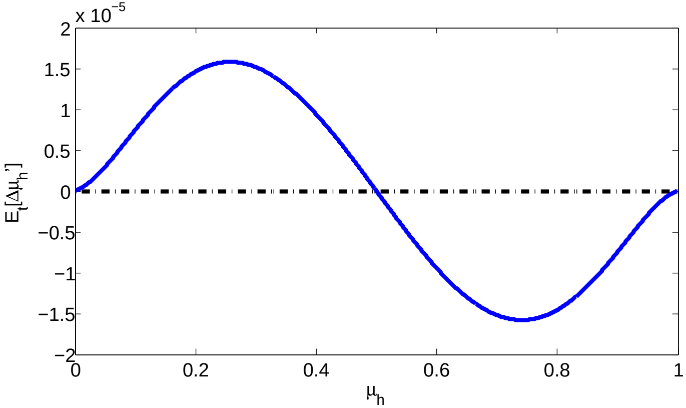

→ $E_t\left[\mu_{h,t+1}\right] > \mu_{h,t}$, if $\mu_{h,t} \leq 1/2$

→ $E_t\left[\mu_{h,t+1}\right] < \mu_{h,t}$, if $\mu_{h,t} > 1/2$

Time-invariant distribution of Pareto weights

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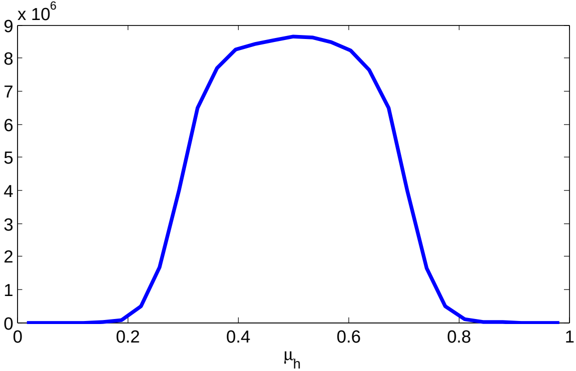

$\longrightarrow$ Ergodic distribution is symmetric around $1/2$

Unscaled Moments (Rare Events)

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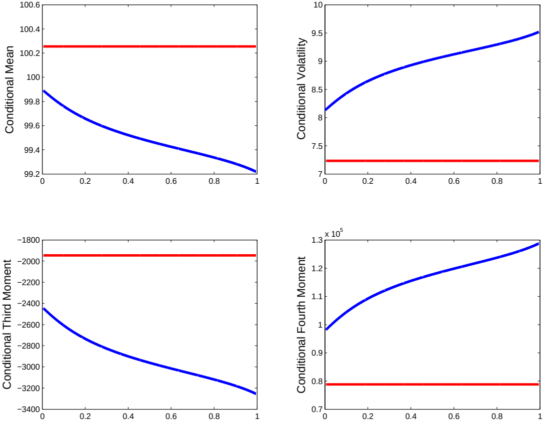
Home Country, Home Good (X)



Rare Events

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X	Y	π
103	103	0.2375
103	100	0.2375
100	103	0.2375
100	100	0.2375
103	60	0.0100
100	60	0.0100
60	60	0.0100
60	103	0.0100
60	100	0.0100

Rare Events

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Four equally likely no-disaster events →

X	Y	π
103	103	0.2375
103	100	0.2375
100	103	0.2375
100	100	0.2375
103	60	0.0100
100	60	0.0100
60	60	0.0100
60	103	0.0100
60	100	0.0100

Rare Events

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X	Y	π
103	103	0.2375
103	100	0.2375
100	103	0.2375
100	100	0.2375
103	60	0.0100
100	60	0.0100
60	60	0.0100
60	103	0.0100
60	100	0.0100

Five equally likely disaster events