The Term Structures of Co-Entropy in International Financial Markets

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Motivation

- One of the best established observations in international finance: int'l stochastic discount factors should be highly correlated
- Key for:
  - high correlation of stock mkt returns (despite low correlation of fundamentals)
  - relative smoothness of FX (relative to high vol of SDF's)
  - ...

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- Key for:
  - high correlation of stock mkt returns (despite low correlation of fundamentals)
  - relative smoothness of FX (relative to high vol of SDF's)
  - ...
- Several int'l macro-finance model can account for this: Colacito and Croce (JPE, 2011); Lustig, Roussanov, and Verdelhan (RFS, 2011, JFE 2012); Stathopoulos (2013); Fahri and Gabaix (2013); ...

Raising the bar

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2. Break-down total co-entropy into permanent and transitory components
3. Look at co-entropies at multiple horizons
4. Confront models with a richer set of over-identifying restrictions

What do we find?

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1. Co-entropy of **total** SDF is high
2. Co-entropy of **permanent** SDF is high
3. Co-entropy of **transitory** components is:
    - low at short horizons
    - high at long horizons
4. No existing macro-finance model can account for this

Where does this leave us?

- A rich set of identifying restrictions for international macro-finance models
- Models are usually focused on the **contemporaneous** correlation of shocks across countries
- We need to think harder about the **inter-temporal** correlation of shocks across countries

Related Literature

- **Entropy bounds**: Bansal and Lehmann (MD, 1997); Backus, Chernov, and Zin (JF, 2013); Bakshi and Chabi-Yo (JFE, 2012)
- **Decomposition of SDF**: Bansal and Lehmann (MD, 1997); Alvarez and Jermann (Ecta, 2005); Hansen (Ecta, 2012); Hansen and Scheinkman (Ecta, 2009).
- **International Macro-Finance**: Brandt, Cochrane, and Santa-Clara (JME, 2006); Verdelhan (JF, 2010); Colacito and Croce (JPE, 2011); Lustig, Stathopulos, and Verdelhan (WP, 2014); Lustig, Roussanov, and Verdelhan (RFS, 2011).

What is Co-Entropy?

- A generalized measure of correlation

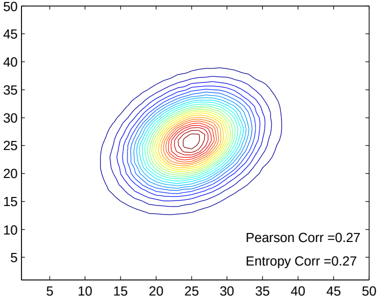
$$\rho_{M^*,M} = 1 - \frac{L[M^*/M]}{L[M] + L[M^*]}$$

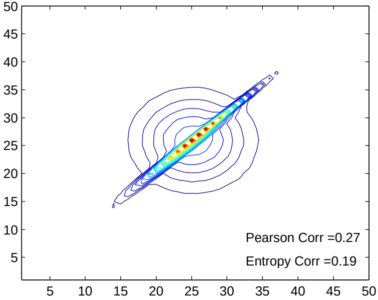
where

$$L[X] = \log E[X] - E[\log(X)]$$

- If SDF's are log-normally distributed: co-entropy is plain-vanilla correlation.

An example

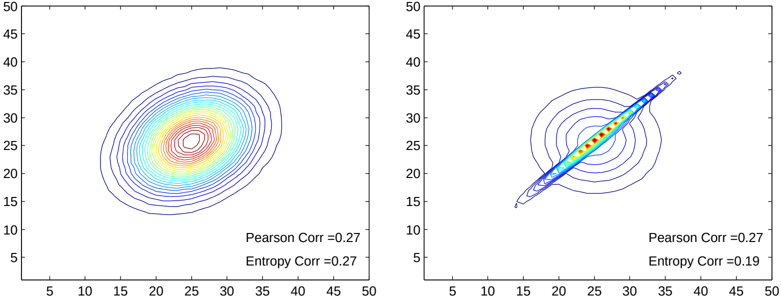




$\rightarrow$ Two log-normal distributions

$\rightarrow$ Mixture of normals

An example

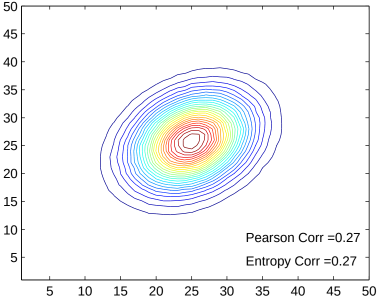


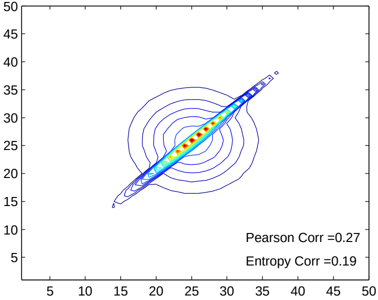
→ Two log-normal distributions

→ Identical correlation measures

→ Mixture of normals

An example





→ Two log-normal distributions
→ Identical correlation measures

→ Mixture of normals
→ Co-entropy is more conservative

Lower Bound on the Co-Entropy of SDF's

$$\rho_{M^*,M} \geq 1 - \frac{L[\exp(\Delta e)]}{E[r_{ex}]+E[r^*_{ex}]}$$

More

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More

- Co-Entropy is close to 1 if:
  1. FX is smooth
  2. Equity risk premia are large

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More

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  1. FX is smooth
  2. Equity risk premia are large
- Data?

Lower Bound on the Co-Entropy of SDF's (cont'd)

The lower bound is close to 1

	1	3	12	24	36	48	Slope
UK	0.96	0.95	0.93	0.94	0.94	0.95	−0.01
	(0.88, 1.00)	(0.73, 1.00)	(0.77, 1.00)	(0.77, 1.00)	(0.78, 1.00)	(0.79, 1.00)	[0.657]
CAN	0.99	0.98	0.98	0.98	0.98	0.97	−0.01
	(0.93, 1.00)	(0.90, 1.00)	(0.89, 1.00)	(0.92, 1.00)	(0.91, 1.00)	(0.91, 1.00)	[0.747]
JPN	0.94	0.91	0.88	0.86	0.85	0.85	−0.03
	(0.68, 1.00)	(0.55, 1.00)	(0.31, 1.00)	(0.23, 1.00)	(0.23, 1.00)	(0.27, 1.00)	[0.610]
FRA	0.96	0.95	0.94	0.93	0.93	0.94	−0.02
	(0.83, 1.00)	(0.71, 1.00)	(0.71, 1.00)	(0.68, 1.00)	(0.69, 1.00)	(0.72, 1.00)	[0.671]
GER	0.96	0.95	0.94	0.93	0.93	0.93	−0.02
	(0.86, 1.00)	(0.76, 1.00)	(0.76, 1.00)	(0.71, 1.00)	(0.69, 1.00)	(0.70, 1.00)	[0.687]
ITA	0.95	0.93	0.92	0.92	0.92	0.93	−0.02
	(0.72, 1.00)	(0.61, 1.00)	(0.58, 1.00)	(0.59, 1.00)	(0.61, 1.00)	(0.61, 1.00)	[0.611]
AUT	0.94	0.93	0.90	0.91	0.92	0.92	−0.01
	(0.71, 1.00)	(0.67, 1.00)	(0.51, 1.00)	(0.53, 1.00)	(0.56, 1.00)	(0.55, 1.00)	[0.574]
BEL	0.96	0.95	0.94	0.93	0.93	0.93	−0.03
	(0.85, 1.00)	(0.72, 1.00)	(0.74, 1.00)	(0.70, 1.00)	(0.72, 1.00)	(0.73, 1.00)	[0.707]
DEN	0.96	0.94	0.93	0.93	0.93	0.93	−0.02
	(0.77, 1.00)	(0.69, 1.00)	(0.66, 1.00)	(0.63, 1.00)	(0.64, 1.00)	(0.67, 1.00)	[0.623]
FIN	0.96	0.95	0.95	0.95	0.95	0.95	−0.01
	(0.77, 1.00)	(0.74, 1.00)	(0.78, 1.00)	(0.73, 1.00)	(0.73, 1.00)	(0.69, 1.00)	[0.618]
IRE	0.96	0.94	0.92	0.92	0.92	0.87	0.00
	(0.76, 1.00)	(0.73, 1.00)	(0.61, 1.00)	(0.62, 1.00)	(0.60, 1.00)	(−1.00, 1.00)	[0.530]
NED	0.96	0.95	0.94	0.94	0.94	0.95	−0.01
	(0.81, 1.00)	(0.71, 1.00)	(0.72, 1.00)	(0.70, 1.00)	(0.73, 1.00)	(0.74, 1.00)	[0.592]
NOR	0.97	0.95	0.94	0.95	0.95	0.95	−0.01
	(0.84, 1.00)	(0.72, 1.00)	(0.65, 1.00)	(0.68, 1.00)	(0.71, 1.00)	(0.78, 1.00)	[0.615]
SPA	0.96	0.94	0.93	0.92	0.92	0.93	−0.03
	(0.83, 1.00)	(0.71, 1.00)	(0.71, 1.00)	(0.69, 1.00)	(0.65, 1.00)	(0.67, 1.00)	[0.713]
SWE	0.97	0.95	0.95	0.95	0.95	0.96	−0.01
	(0.87, 1.00)	(0.76, 1.00)	(0.79, 1.00)	(0.84, 1.00)	(0.85, 1.00)	(0.86, 1.00)	[0.643]
SUI	0.95	0.94	0.93	0.93	0.94	0.95	−0.00
	(0.82, 1.00)	(0.67, 1.00)	(0.70, 1.00)	(0.72, 1.00)	(0.76, 1.00)	(0.79, 1.00)	[0.504]

Decomposition

- Bansal and Lehmann (1997) and Alvarez and Jermann (2005):

$$M = M^P \cdot M^T$$

  - M^P is the growth rate of the permanent component (a martingale)
  - M^T is the growth rate of the transitory component
  - Measure M^T as the inverse of the return on infinity maturity bond (R^∞)

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- Co-entropy of total SDF's can be decomposed as

$$\rho_{M^*,M} = \alpha_0 + \alpha_1 \cdot \rho_{M^{P*},M^P} + \alpha_2 \cdot \rho_{M^T,M^{*T}} + \alpha_3 \cdot \rho_{\frac{M^{P*}}{M^P},\frac{M^T}{M^{T*}}}$$

where α's are entropy shares.

Measurement of Co-entropy

- Lower bound

$$\rho_{M^{*P},M^{P}} \geq 1 - \frac{L[\exp(\overbrace{\Delta e + r^*_\infty}^{\text{foreign return in domestic units}} - \overbrace{r_\infty}^{\text{domestic return in domestic units}})]}{E[\underbrace{r_{ex,\infty}}_{r_\infty - r_f}] + E[\underbrace{r^*_{ex,\infty}}_{r^*_\infty - r^*_f}]}$$

US & UK

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- Directly observable

$$\rho_{M^T,M^{*T}} = 1 - \frac{L[\exp(\overbrace{r^*_\infty}^{\text{foreign return in foreign units}} - \overbrace{r_\infty}^{\text{domestic return in domestic units}})]}{L[\exp(-r_\infty)] + L[\exp(-r^*_\infty)]}$$

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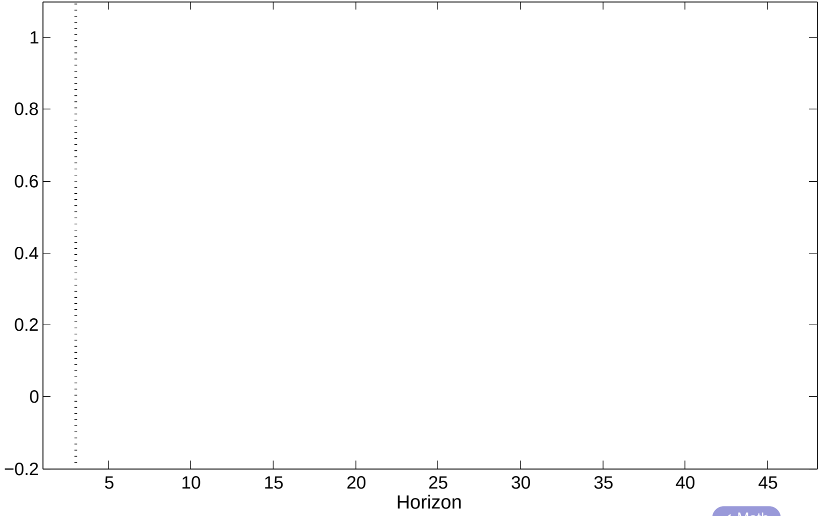
$$\rho_{\frac{M^{*P}}{M^{P}},\frac{M^{T}}{M^{*T}}} = 1-\frac{L[\exp(\Delta e)]}{L[\exp(\Delta e+r_{\infty}^{*}-r_{\infty})]+L[\exp(r_{\infty}^{*}-r_{\infty})]}$$

US & UK

Data

- Large cross section of developed countries.
- Sample: January 1975 – May 2013.
- Stock market returns: value-weighted returns in local currency.
- Risk-free rates: three-month interest rates on Government Bills.
- Long-term rates: ten-year interest rates on Government Bonds.
- CPI inflation: growth rate of the “Total Items” index in consecutive months.
- Exchange rates: units of foreign currency per US dollar.
- Real variables: nominal variables divided by realized CPI inflation.

An example: US vs UK



Math

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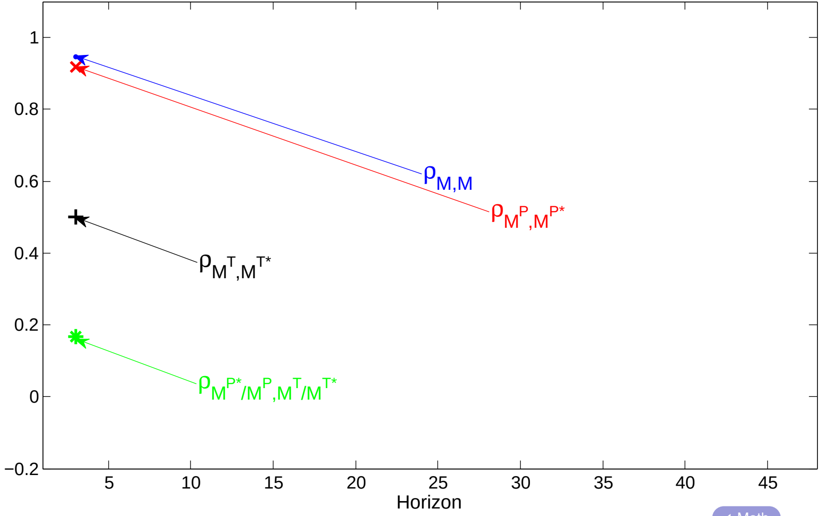

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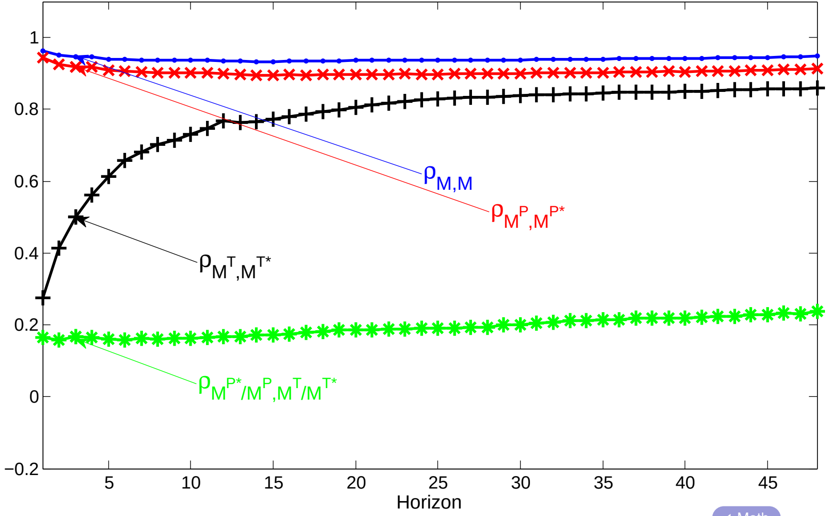

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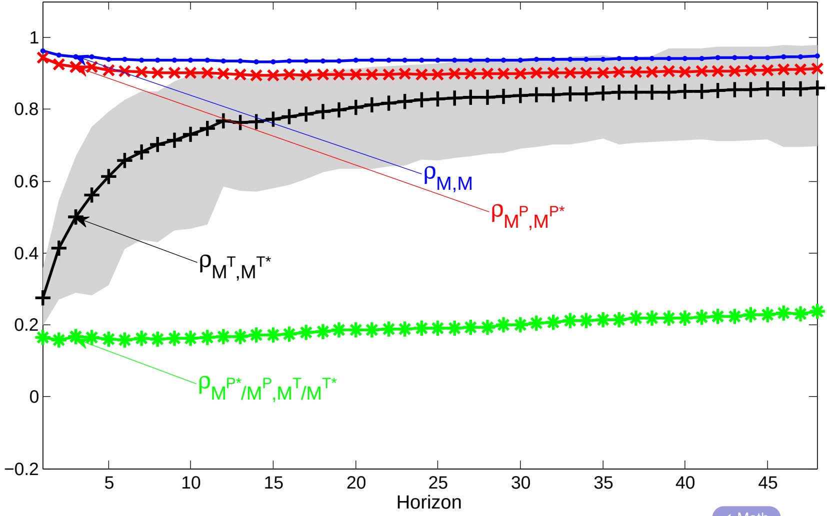
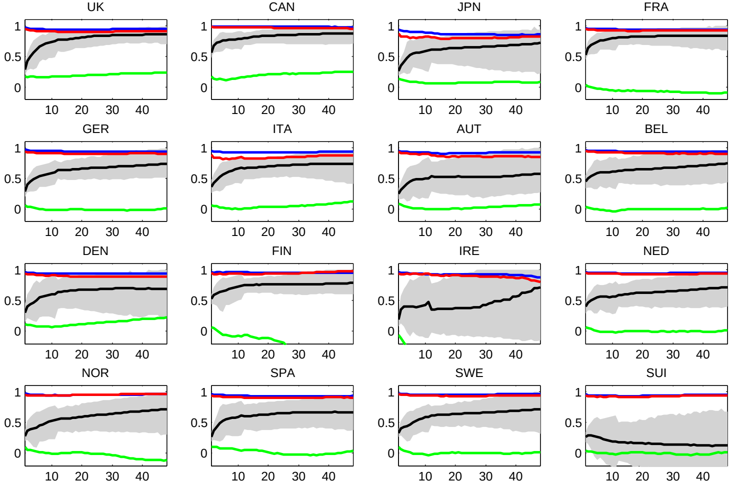


Math

Other countries



Model

Summary of empirical findings

- Co-entropy of
  - Total SDF's is large
  - Permanent components of SDF's is large
  - Transitory components of SDF's
    1. low at short horizons
    2. sharply upward sloping

Summary of empirical findings

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  - Transitory components of SDF's
    1. low at short horizons
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- What's driving them?

  → Int'l correlation of long-term bonds:

  - in domestic units (e.g. all measured in $) is high, no matter the horizon
  - in local units (e.g. measured in $ and £) is
    1. Low at short horizons
    2. High at long horizons

Macro-Finance Models

- Can int'l macro finance models account for these findings?
- We look at several models:
  - the long-run risks model of Colacito and Croce (JPE, 2011)
  - the habits model of Verdelhan (JF, 2010)
  - the rare events model of Barro (QJE, 2006)
  - the reduced form model of Lustig, Roussanov, and Verdelhan (JFE, 2012)
  - ...
- None of them can match simultaneously the term structures of co-entropy.

A long-run risks model

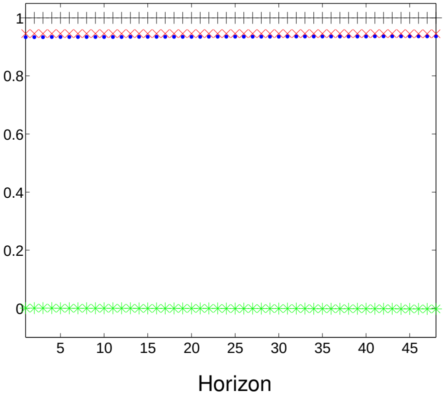
- Model setup

$$
\begin{aligned}
U_t &= (1-\delta)\log C_t + \delta\theta \log E_t \exp\left\{\frac{U_{t+1}}{\theta}\right\} \\
\Delta c_{t+1} &= \mu_c + x_t + \sigma_c \varepsilon_{c,t+1}, \\
x_t &= \rho x_{t-1} + \sigma_x \varepsilon_{x,t},
\end{aligned}
$$

- SDF's and their components

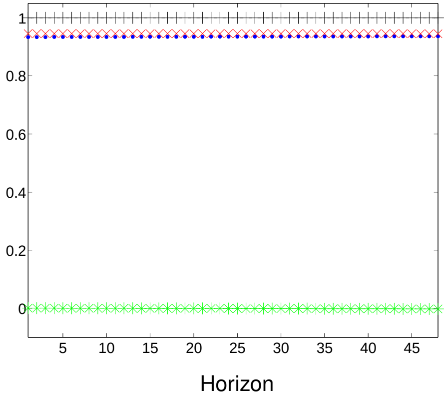
$$
\begin{aligned}
m_{t+1} - E\left[m_{t+1}\right] &= -x_t + \frac{B\sigma_x}{\theta}\varepsilon_{x,t+1} + \left(\frac{1}{\theta}-1\right)\sigma_c\varepsilon_{c,t+1}, \\
m^T_{t+1} - E\left[m^T_{t+1}\right] &= -x_t - \xi\sigma_x\varepsilon_{x,t+1}, \\
m^P_{t+1} - E\left[m^P_{t+1}\right] &= \left(\frac{B}{\theta}+\xi\right)\sigma_x\varepsilon_{x,t+1} + \left(\frac{1}{\theta}-1\right)\sigma_c\varepsilon_{c,t+1}
\end{aligned}
$$

Term Structures of Co-Entropy



Data

Term Structures of Co-Entropy



- $\rho_{M^*,M}$ is high

Data

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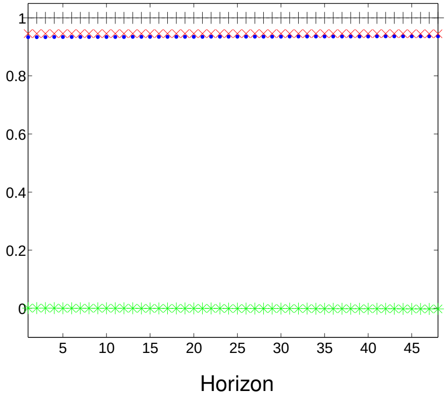
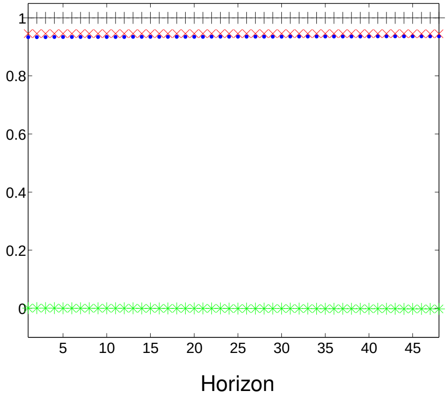


- $\rho_{M^*,M}$ is high ✓

◂ Data

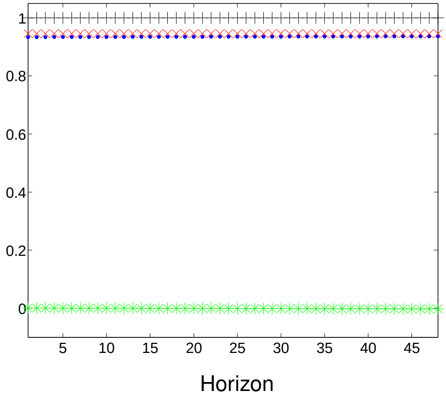
Term Structures of Co-Entropy



- $\rho_{M^*,M}$ is high ✓
- × ρ_{M^{P*},M^P} is high

◂ Data

Term Structures of Co-Entropy



- $\rho_{M^*,M}$ is high ☑
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Data

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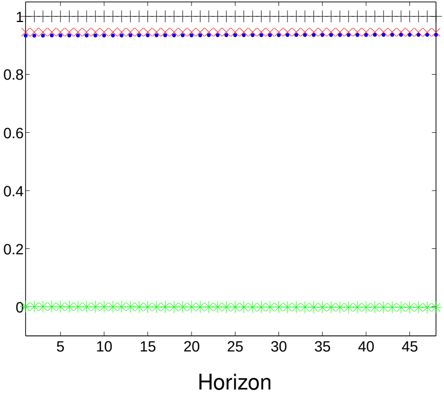
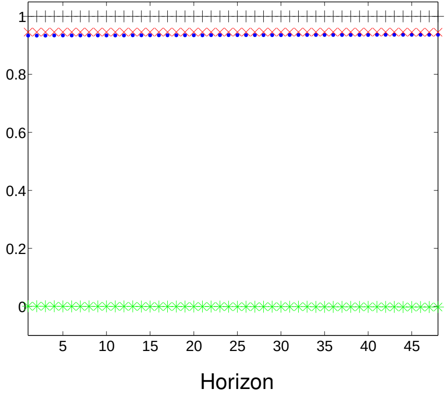


- $\rho_{M^*,M}$ is high ✓
- × ρ_{M^{P*},M^P} is high ✓
- * $\rho_{M^{P*}/M^P, M^T/M^{T*}}$ is low

Data

Term Structures of Co-Entropy



- ● $\rho_{M^*,M}$ is high ☑
- × ρ_{M^{P*},M^P} is high ☑
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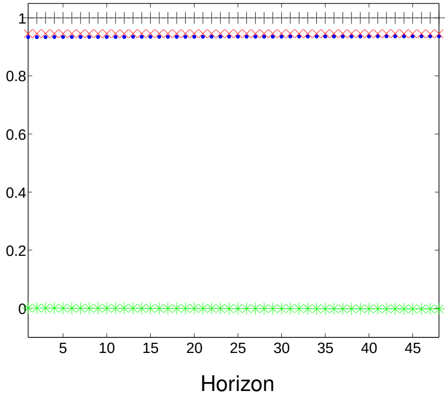
◂ Data

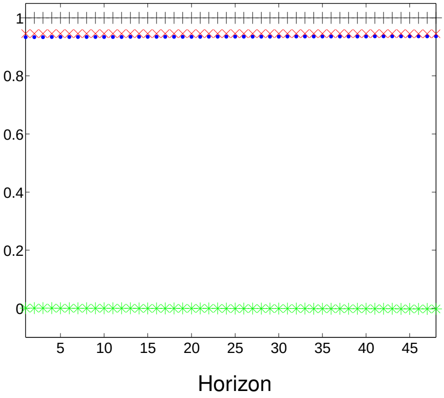
Term Structures of Co-Entropy



- • $\rho_{M^*,M}$ is high ✓
- × ρ_{M^{P*},M^P} is high ✓
- ∗ $\rho_{M^{P*}/M^P,M^T/M^{T*}}$ is low ✓
- + ρ_{M^{T*},M^T} is high and flat

◂ Data

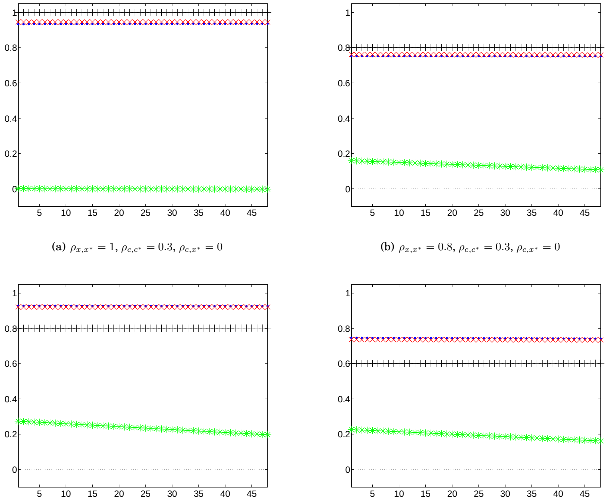
Term Structures of Co-Entropy



- • $\rho_{M^*,M}$ is high ☑
- × ρ_{M^{P*},M^P} is high ☑
- ∗ $\rho_{M^{P*}/M^P,M^T/M^{T*}}$ is low ☑
- + ρ_{M^{T*},M^T} is high and flat ☒

Data

Changing Calibrations: same story...



(a) $\rho_{x,x^*} = 1$, $\rho_{c,c^*} = 0.3$, $\rho_{c,x^*} = 0$

(b) $\rho_{x,x^*} = 0.8$, $\rho_{c,c^*} = 0.3$, $\rho_{c,x^*} = 0$

(c) $\rho_{x,x^*} = 0.8$, $\rho_{c,c^*} = 0.3$, $\rho_{c,x^*} = 0.3$

(d) $\rho_{x,x^*} = 0.6$, $\rho_{c,c^*} = 0.30$, $\rho_{c,x^*} = 0.30$

Concluding Remarks

What we learned:

- A novel measure of correlation (co-entropy)
- A formal decomposition of the sources of int'l co-entropy
- Shapes of term-structures of co-entropy are very robust in the cross-section of countries

What we would like to learn:

- Is there a model that can explain all of these moments?
- We need to think harder about the int'l transmission of shocks across countries and across dates

Back-up Slides

Details on Coentropy bound

- Consider:

$$\rho_{M^*,M} = 1 - \frac{L[M^*/M]}{L[M] + L[M^*]}$$

Back

Details on Coentropy bound

- Consider:

$$\rho_{M^*,M} = 1 - \frac{L\left[M^*/M\right]}{L[M] + L[M^*]}$$

- By no-arbitrage $\exp(\Delta e) = M^*/M$:

$$\rho_{M^*,M} = 1 - \frac{L\left[\exp(\Delta e)\right]}{L[M] + L[M^*]}$$

Back

Details on Coentropy bound

- Consider:

$$\rho_{M^*,M} = 1 - \frac{L[M^*/M]}{L[M] + L[M^*]}$$

- By no-arbitrage $\exp(\Delta e) = M^*/M$:

$$\rho_{M^*,M} = 1 - \frac{L[\exp(\Delta e)]}{L[M] + L[M^*]}$$

- Bansal and Lehmann's entropy bound $L[M] \geq E[r_{ex}]$:

$$\rho_{M^*,M} \geq 1 - \frac{L[\exp(\Delta e)]}{E[r_{ex}] + E[r^*_{ex}]}$$

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Details on Coentropy bound (M^P)

- Consider:

$$\rho_{M^{P*},M^P} = 1 - \frac{L\left[M^*/M \cdot M^T/M^{T*}\right]}{L[M^P] + L[M^{P*}]}$$

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- Since $M^T = 1/R_\infty$:

$$\rho_{M^{P*},M^P} \geq 1 - \frac{L[\exp(\Delta e + r^*_\infty - r_\infty)]}{E[r_{ex,\infty}] + E[r^*_{ex,\infty}]}$$

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Details on Coentropy bound (M^T)

- Consider:

$$\rho_{M^T,M^{T*}} = 1 - \frac{L\left[M^T/M^{T*}\right]}{L[M^T] + L[M^{T*}]}$$

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- Consider:

$$\rho_{M^T,M^{T*}} = 1 - \frac{L\left[M^T / M^{T*}\right]}{L[M^T] + L[M^{T*}]}$$

- Since $M^T = 1/R_\infty$:

$$\rho_{M^T,M^{T*}} = 1 - \frac{L\left[\exp(r_\infty^* - r_\infty)\right]}{L[exp(-r_\infty^*)] + L[exp(-r_\infty)]}$$

Back