Testing and valuing dynamic correlations for asset allocation

Riccardo Colacito and Robert Engle

Goal of the paper

- What is it worth to an investor to have a correct covariance matrix?
- Can these benefits be used to statistically discriminate between covariance matrices with real data?

Our approach

- Each day, minimize portfolio variance subject to a required return, assuming a risk free rate and allowing short positions:

$$\begin{aligned} \min_{w_t} \quad & w_t' H_{t/t-1} w_t \\ s.t. \quad & w_t' \mu_{t/t-1} \geq \mu_0 \end{aligned}$$

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- How can we evaluate the quality of covariance matrix forecasts without knowing expected returns?

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3. They test the joint hypothesis of correct specification of mean and variance.
4. We use constant expected returns and repeat the analysis for a number of possible vectors.

Outline of the talk

- Proposed strategy.
- One way of estimating covariance matrices: Dynamic Conditional Correlation (DCC).
- Results: in sample and simulations.
- More advanced questions and ongoing research.

Solution

- The solution is

$$w_t = \frac{H_t^{-1}\mu}{\mu' H_t^{-1}\mu}\mu_0$$

- This solution always exists provided that H_t is positive definite and the required returns is nonnegative.
- But suppose that H_t is not the true covariance matrix...

Solution

- If Ω_t is the true covariance, the minimized volatility is

$$\frac{\sigma_t^H}{\mu_0} = \frac{\sqrt{\mu' H_t^{-1} \Omega_t H_t^{-1} \mu}}{\mu' H_t^{-1} \mu}$$

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Value of covariance information

- The investor with the correct covariance matrix can achieve the same volatility and a higher required return. Setting volatilities equal:

$$\frac{\mu_0^{\Omega}}{\mu_0^{H}} = \frac{\sqrt{\left(\mu' H_t^{-1} \Omega_t H_t^{-1} \mu\right)\left(\mu' \Omega_t^{-1} \mu\right)}}{\mu' H_t^{-1} \mu} \geq 1$$

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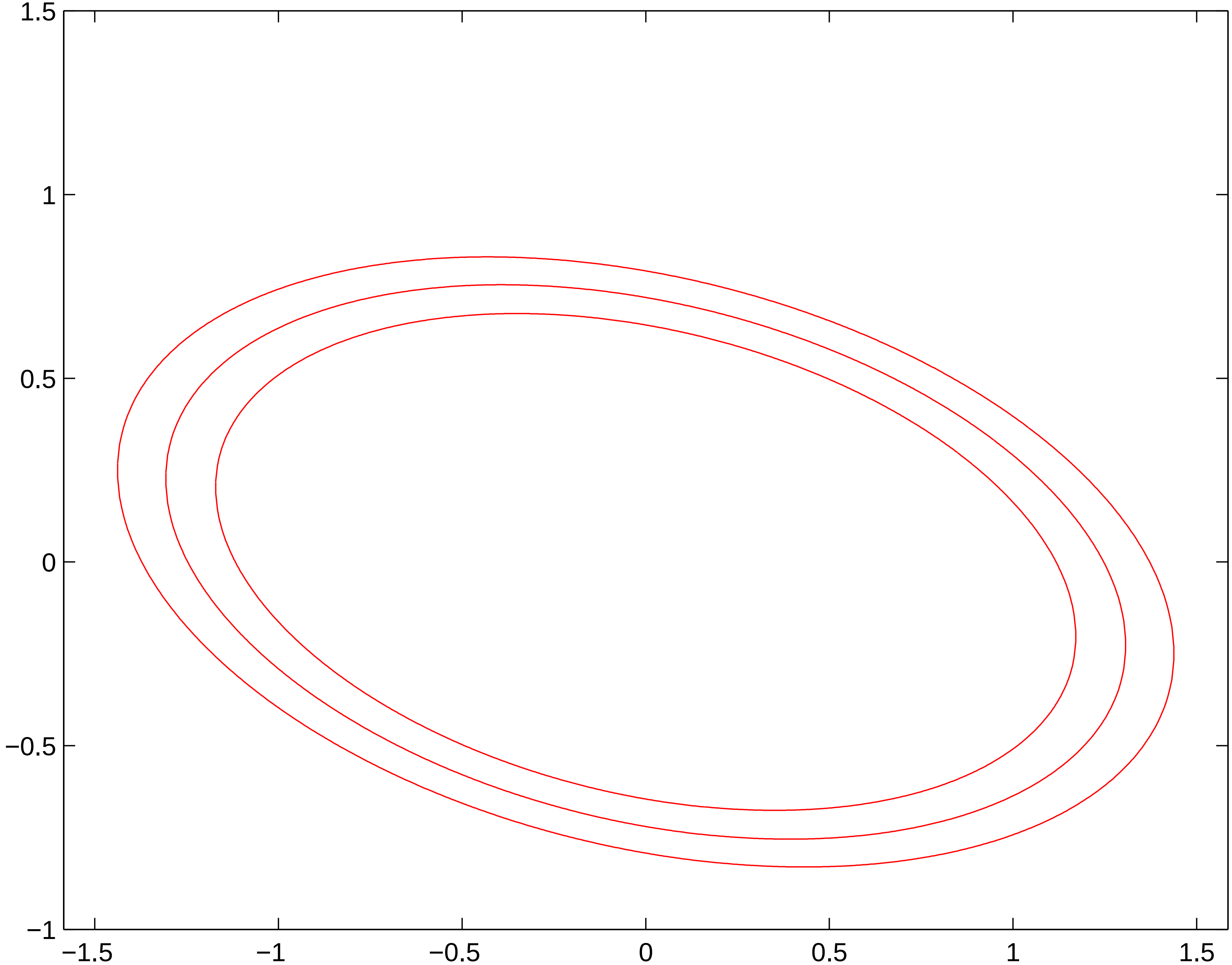
- The ratio of required excess returns giving equal volatility is always larger than 1 for any vector of expected returns.
- Gains will depend upon the choice of μ.
- A costless mistake: if μ is an eigenvector of ΩH^{-1} using the wrong covariance matrix is costless.

Bivariate Example



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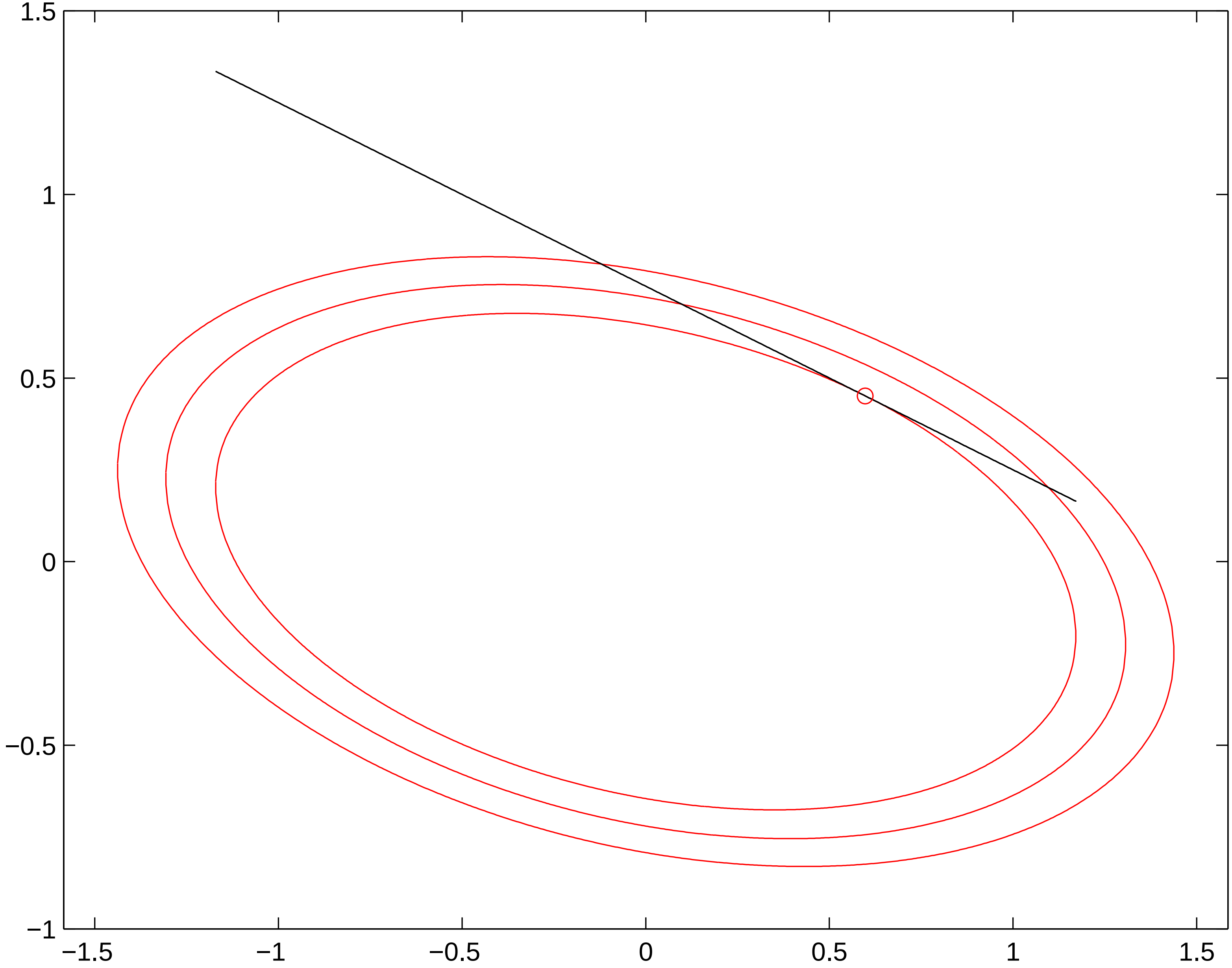

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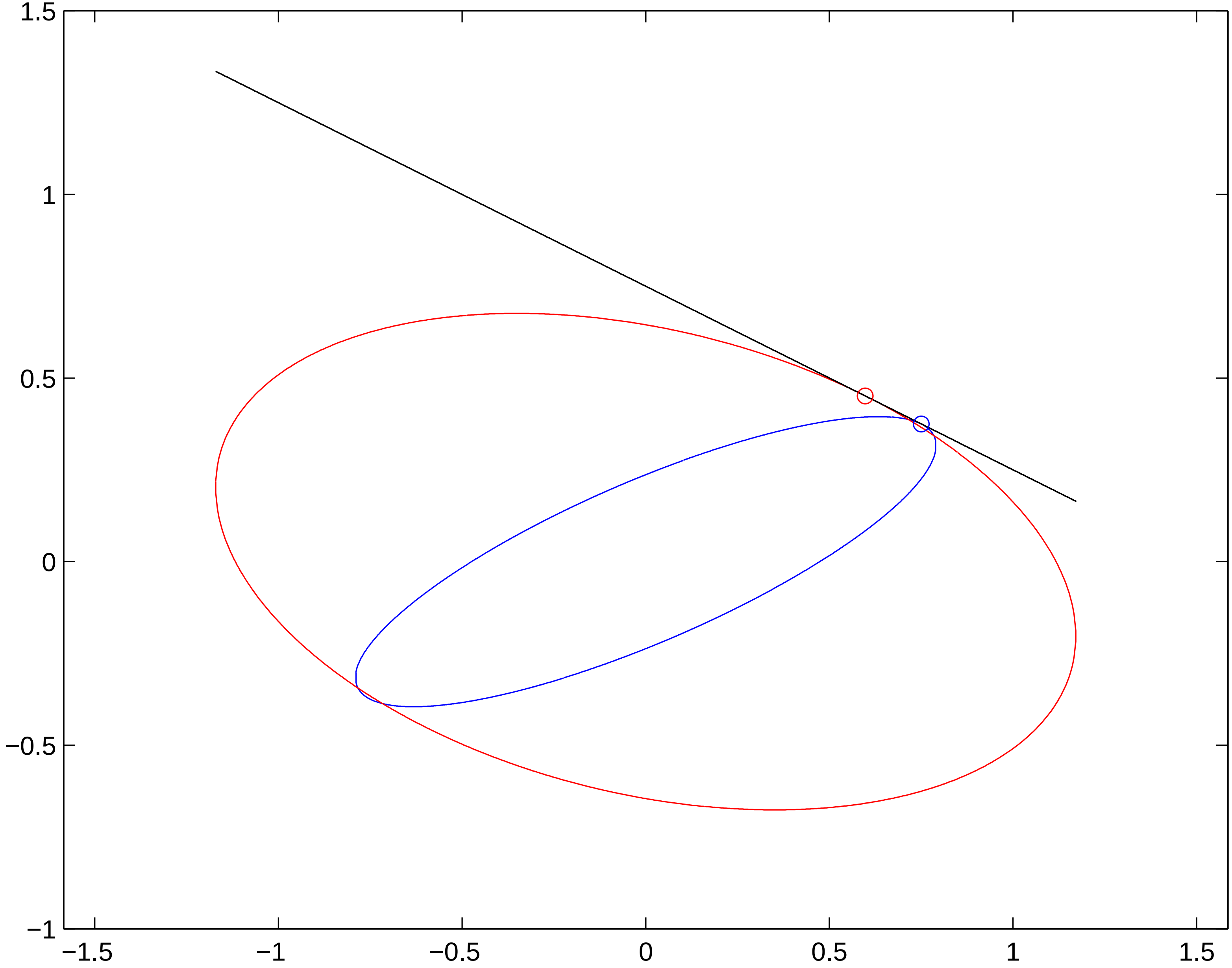

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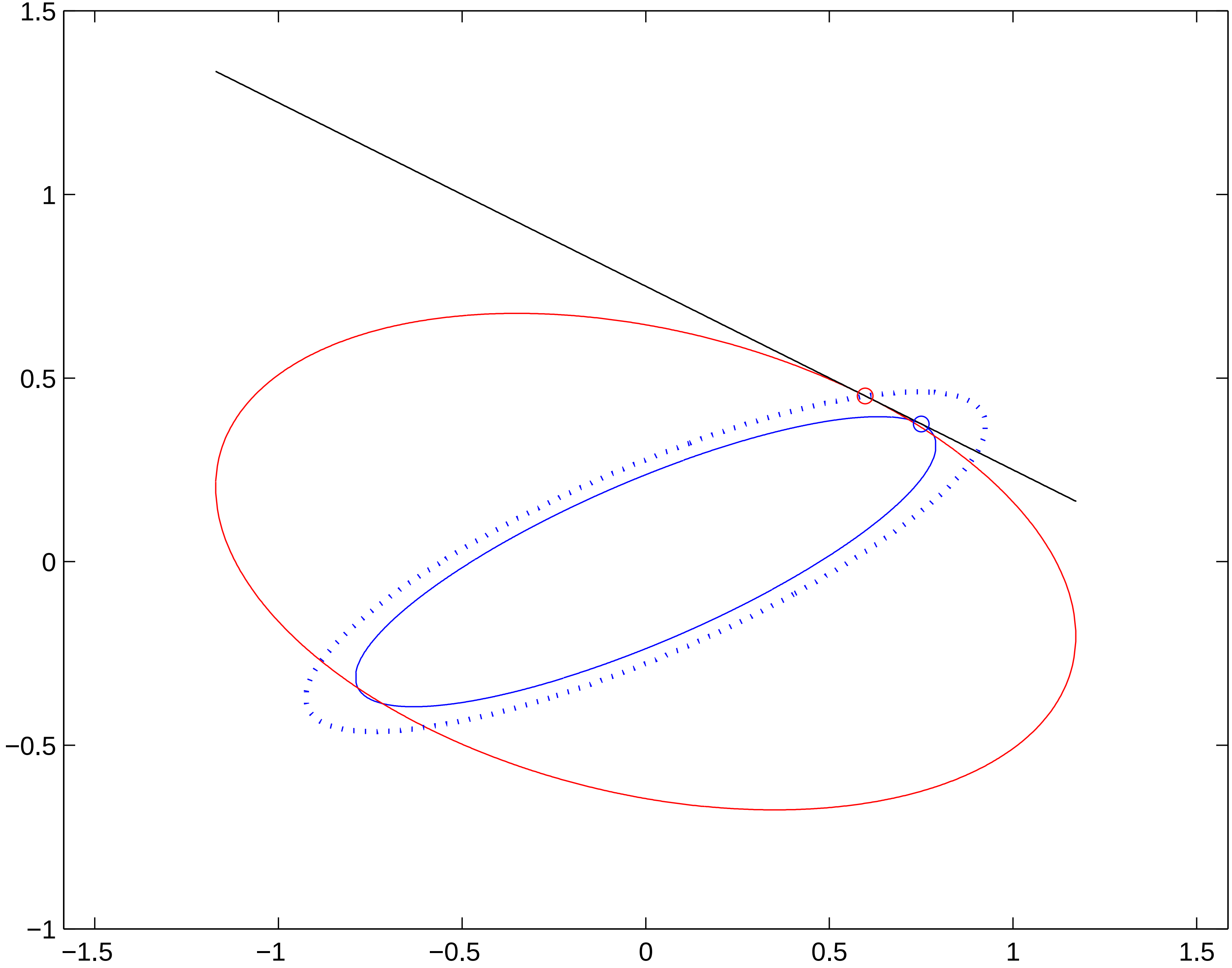
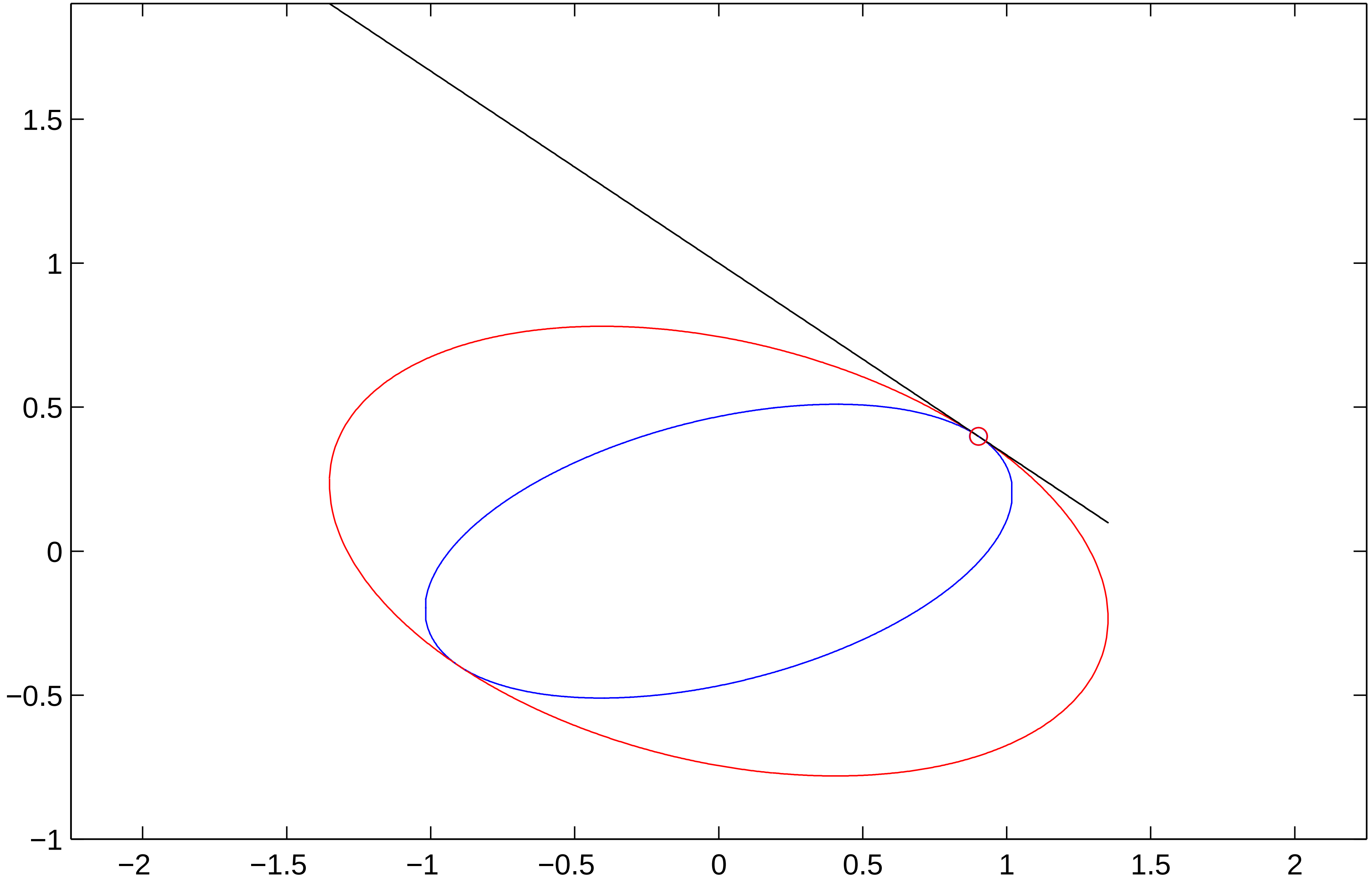


A costless mistake



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- For a vector of expected returns, and a conditional covariance matrix, calculate the optimal weights and the subsequent portfolio return.

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- For a vector of expected returns, and a conditional covariance matrix, calculate the optimal weights and the subsequent portfolio return.
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- Use the approach of Diebold and Mariano (1995) to test that a method is significantly better than another.

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- We also consider a weighted version of the test

$$\frac{\left(w'_{1,t} r_t\right)^2 - \left(w'_{2,t} r_t\right)^2}{\sqrt{\left(\mu' H_{1,t}^{-1} \mu\right)\left(\mu' H_{2,t}^{-1} \mu\right)}} = \xi + \varepsilon_{k,v,t}$$

Joint test of equality of two models

- Stack differences into vectors

$$\begin{aligned} U_t &= (u_{1,t}, ..., u_{K,t})' \\ V_t &= (v_{1,t}, ..., v_{K,t})' \end{aligned}$$

- Use GMM with vector HAC to estimate

$$\begin{aligned} U_t &= \beta_u \iota + \varepsilon_{u,t} \\ V_t &= \beta_v \iota + \varepsilon_{v,t} \end{aligned}$$

- Under the null β_u and β_v are both equal to zero.
- If the null is rejected we can see which way it is rejected.

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- If $r_{i,t} = \sqrt{h_{i,t}} \varepsilon_{i,t}$, with $E_{t-1}[h_{i,t}] = h_{i,t}$, $\forall i = 1, 2$

$$\rho_t = \frac{E_{t-1}[\varepsilon_{1,t} \varepsilon_{2,t}]}{\sqrt{E_{t-1}[\varepsilon_{1,t}^2] E_{t-1}[\varepsilon_{2,t}^2]}}$$

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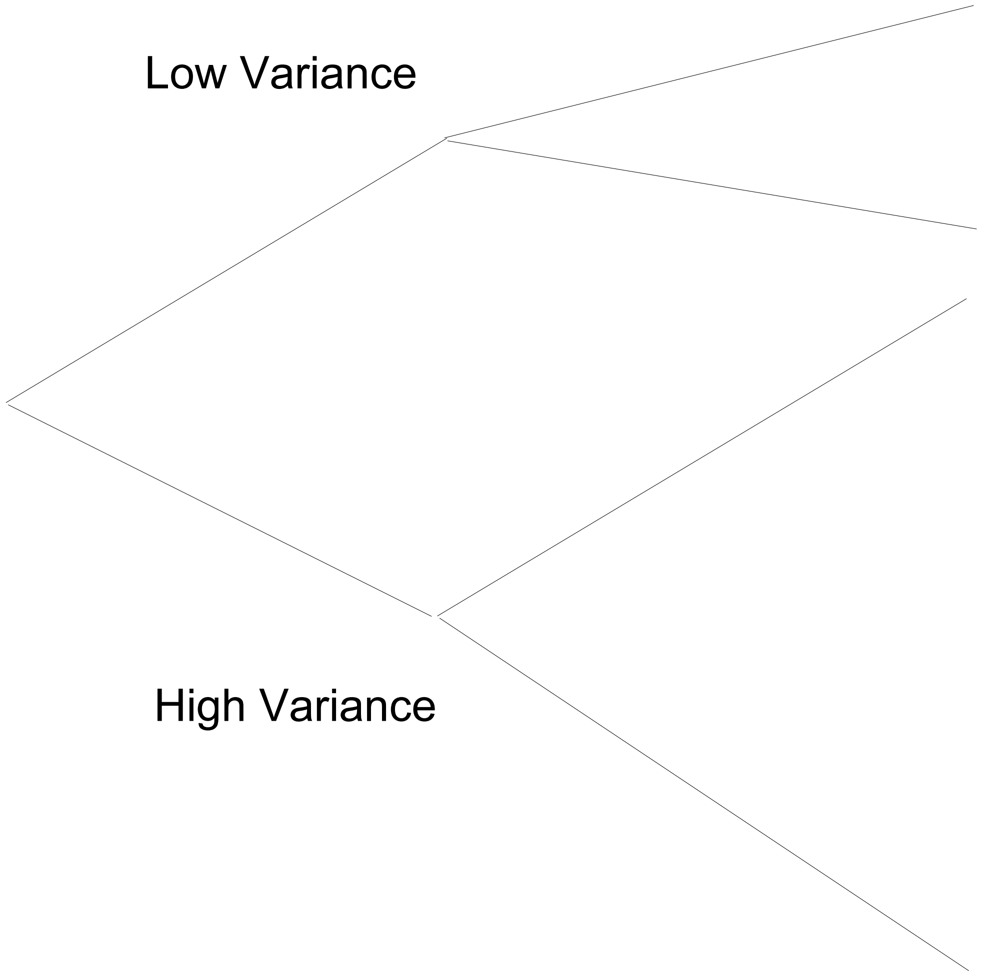
   - E.g. Mean reverting DCC:

$$
\begin{aligned}
Q_t &= \overline{R}(1-\theta_1-\theta_2) + \theta_1 Q_{t-1} + \theta_2 \varepsilon'_t \varepsilon_t \\
R_t &= diag(Q_t)^{-\frac{1}{2}} Q_t diag(Q_t)^{-\frac{1}{2}} \\
L &= -\frac{1}{2}\sum \left[\log|R_t| + \varepsilon'_t R_t \varepsilon_t\right]
\end{aligned}
$$

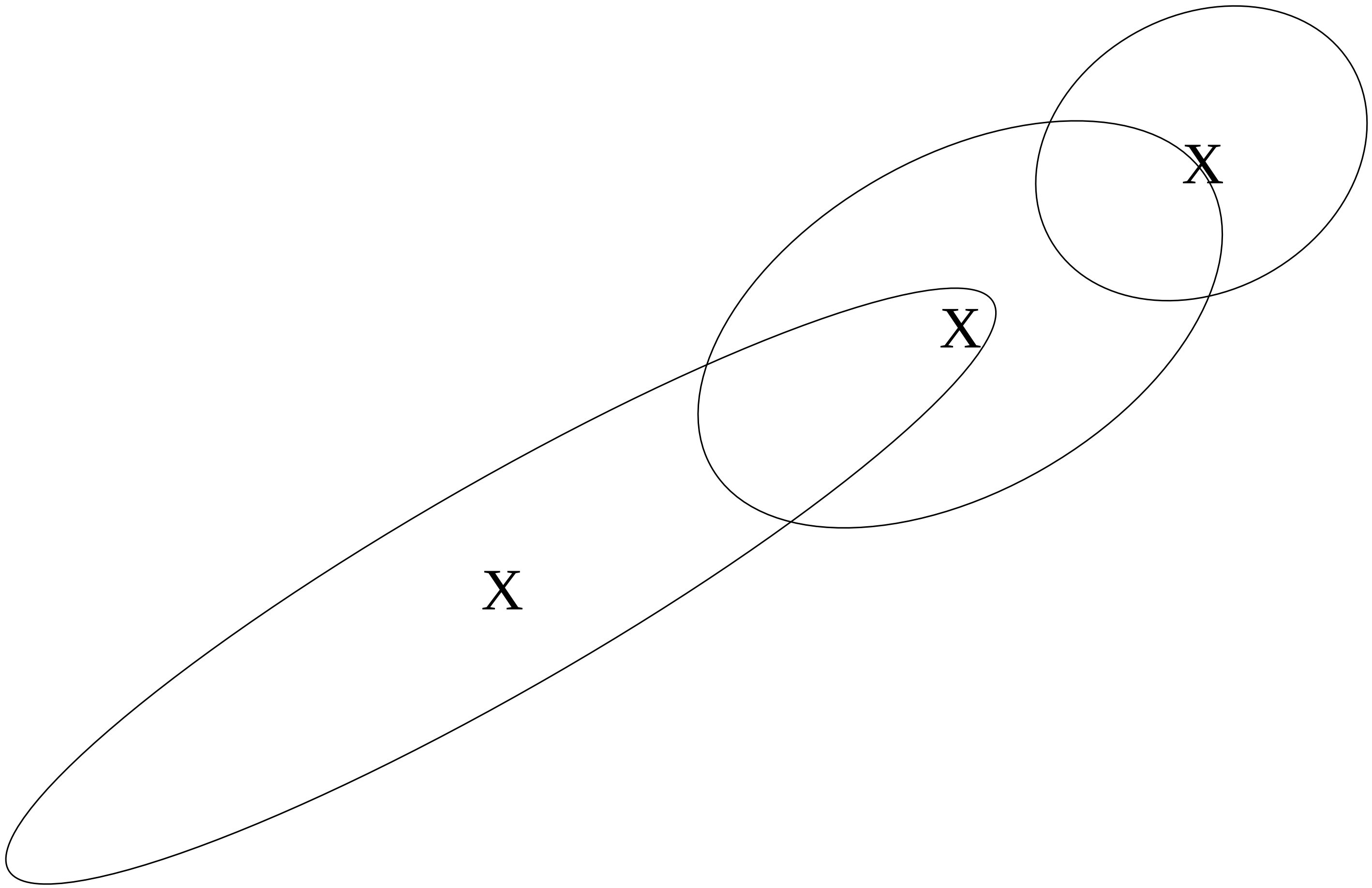
Asymmetric volatilities: intuition

- Engle and Ng (1993): asymmetric impact of news on volatility.



Asymmetric correlations: intuition

- Cappiello, Engle and Sheppard (2004): asymmetric correlations to account for lower tail dependence.



Data

- Stocks (S&P500) and Bonds (10 year Treasury Notes) from August 1988 to August 2003.
- Summary statistics

	Stocks	Bonds
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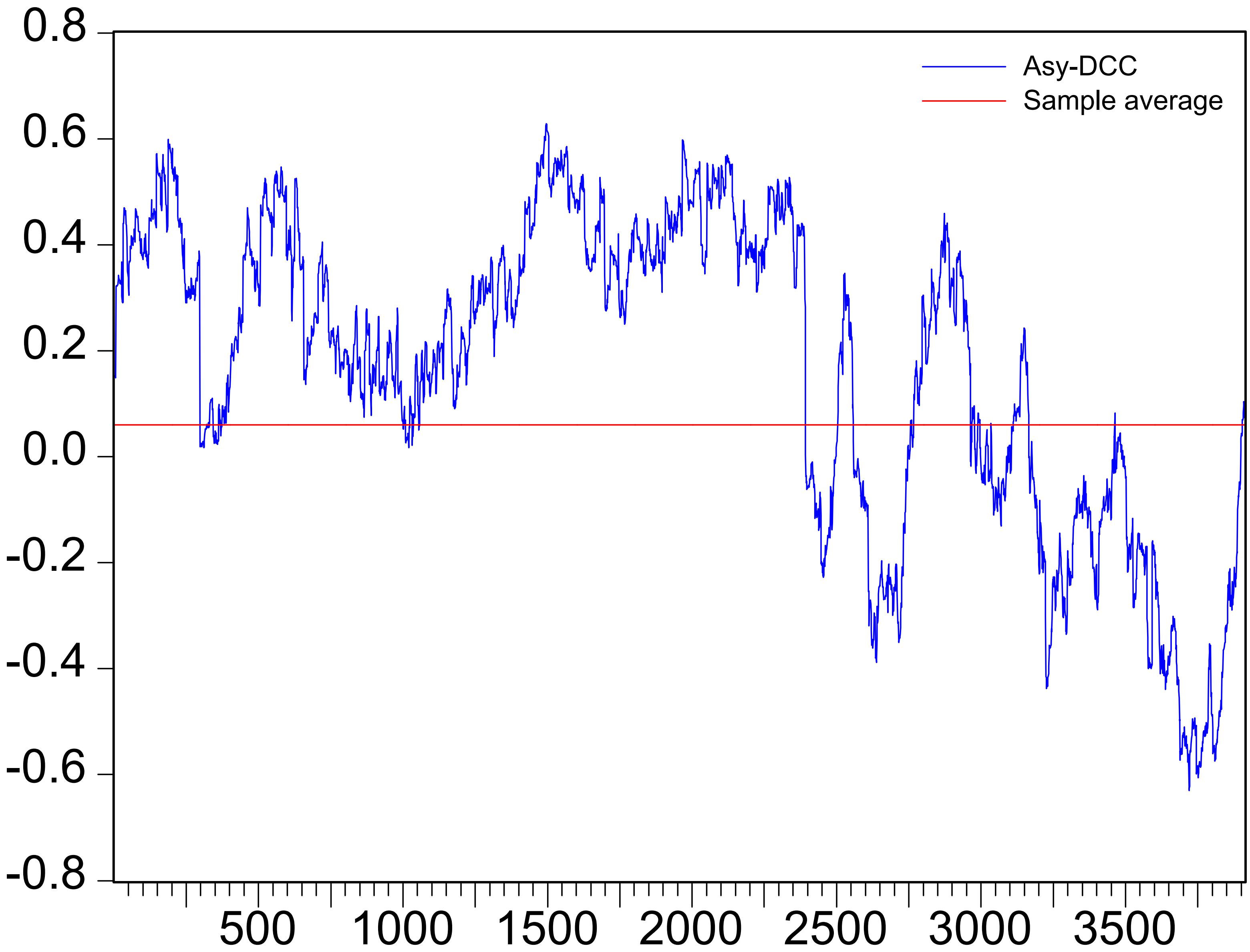
- Average correlation is 0.06.
- Compare two estimators of the covariance matrix:
  1. Constant - unconditional
  2. Asymmetric DCC

Conditional Correlations

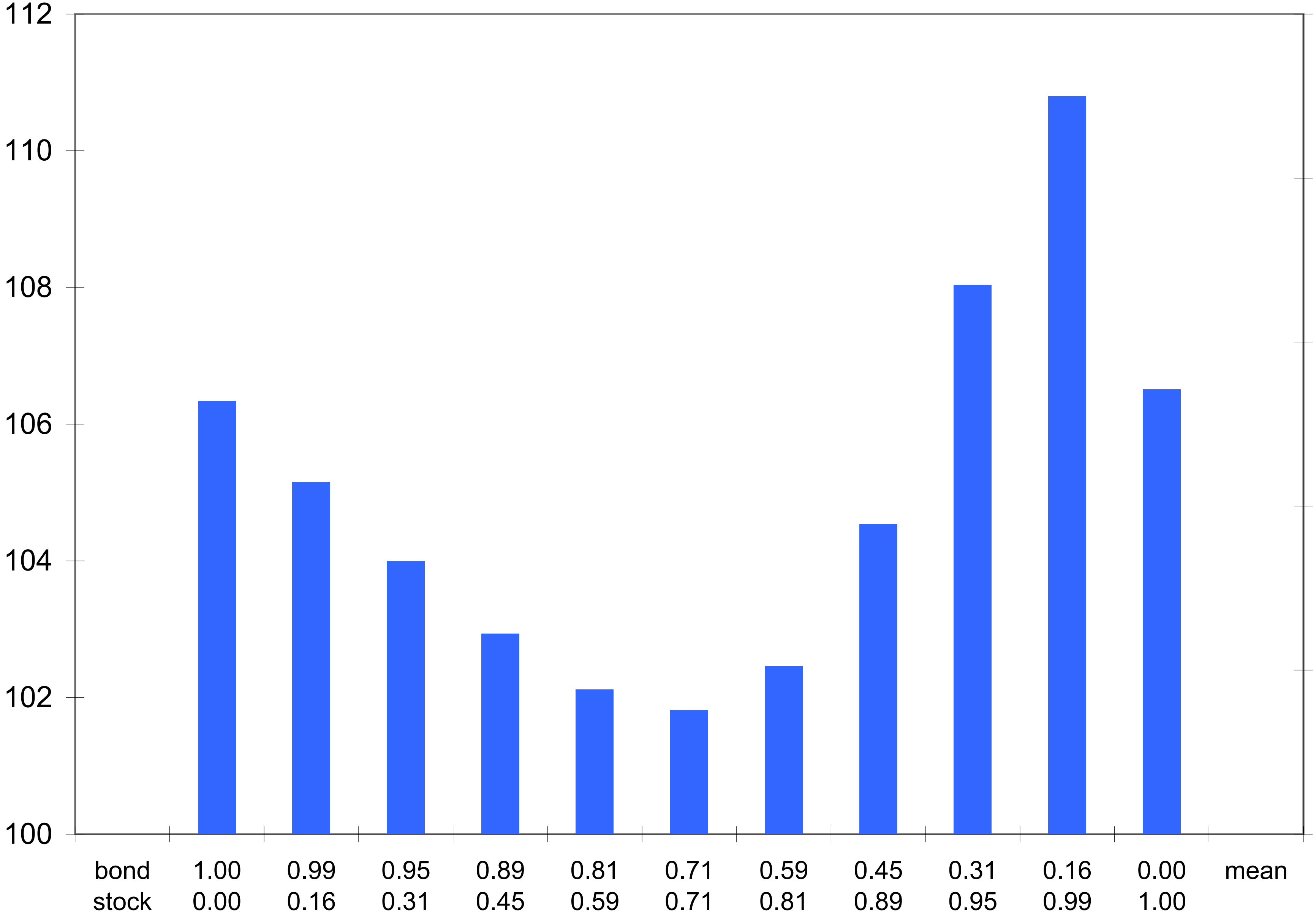


Interpreting results

A number such as 105 means required excess returns are 5% greater with correct correlations.

- E.g. a 4% excess return with incorrect correlation would be a 4.2% return with correct correlations.
- With 10% required return, the value of such correlations is 50 basis points.

Value gains: DCC vs Constant



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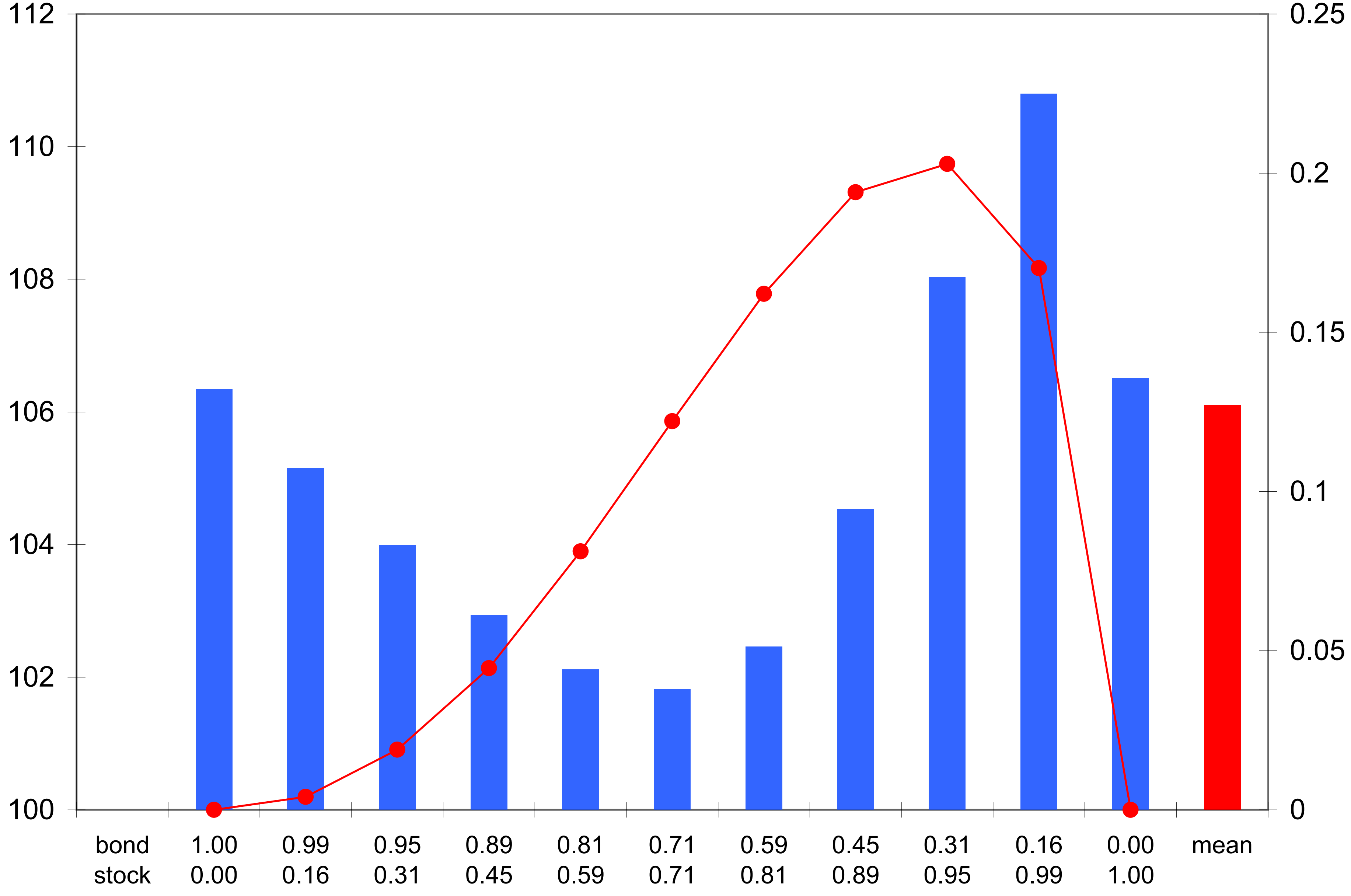


Diebold and Mariano univariate test

- Test for a specific $\mu = [0.16, 0.99]$:

	Sc GARCH	Diag BEKK	DCC-MR	OGARCH	DCC-Asy	Constant
Sc GARCH	-	-0.635	-2.243	13.645	-3.405	7.312
Diag BEKK	0.635	-	-1.278	14.170	-2.764	7.347
DCC-MR	2.543	1.278	-	14.179	-2.470	7.382
OGARCH	-13.645	-14.170	-14.179	-	-14.328	-10.761
DCC-Asy	3.405	2.764	2.470	14.328	-	7.493
Constant	-7.312	-7.347	-7.382	10.761	-7.493	-

Diebold and Mariano joint test

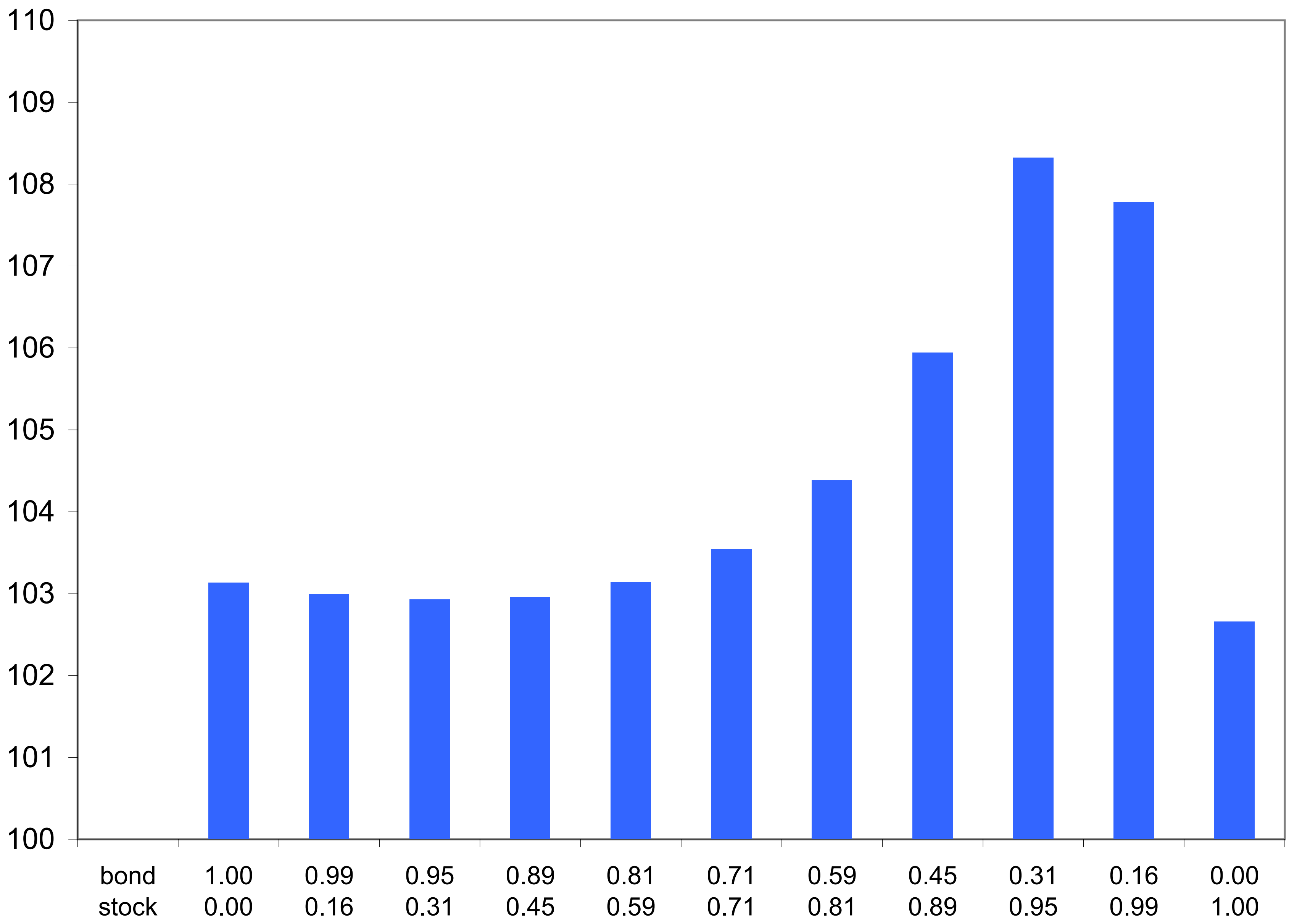
- Test for all vectors of expected returns:

	Sc GARCH	Diag BEKK	DCC-MR	OGARCH	DCC-Asy	Constant
Sc GARCH	-	-3.277	-4.095	12.314	-4.043	5.322
Diag BEKK	3.277	-	-0.427	13.139	-1.299	7.129
DCC-MR	4.095	0.427	-	13.415	0.223	7.049
OGARCH	-12.314	-13.193	-13.415	-	-14.022	-9.696
DCC-Asy	4.043	1.299	-0.223	14.022	-	6.794
Constant	-5.322	-7.129	-7.049	9.696	-6.794	-

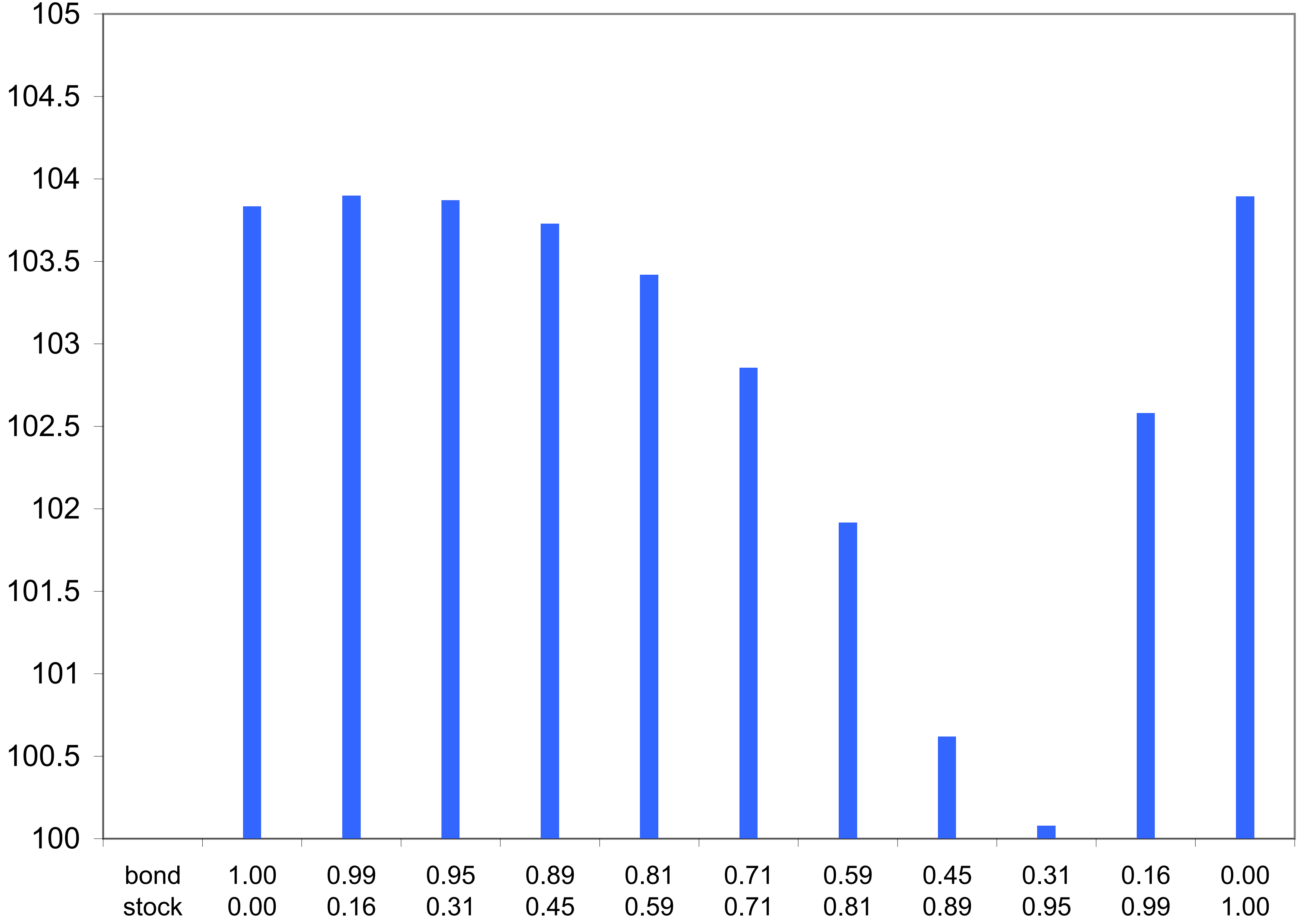
Value gains: DCC vs Constant

- Simulate 10,000 days of the DCC model documented above.
- One investor knows the volatilities and correlations every day, Ω.
- The other only knows the unconditional volatilities and correlations, H.
- What is the gain to the informed investor?

Simulated data (Full Covariance)

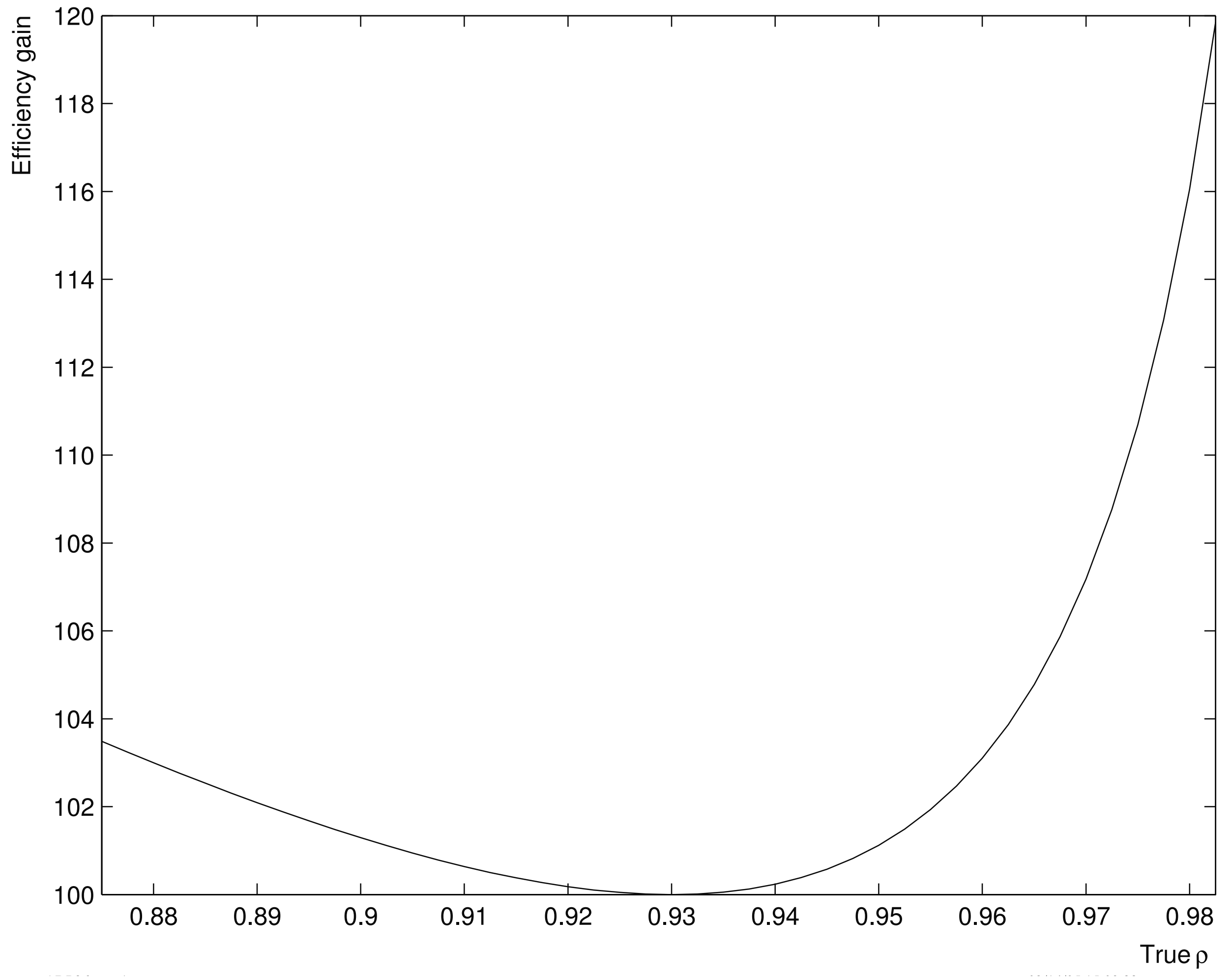


Simulated data (Correlations)



Extreme correlations

- The value of right correlation information is high when correlations are extreme.



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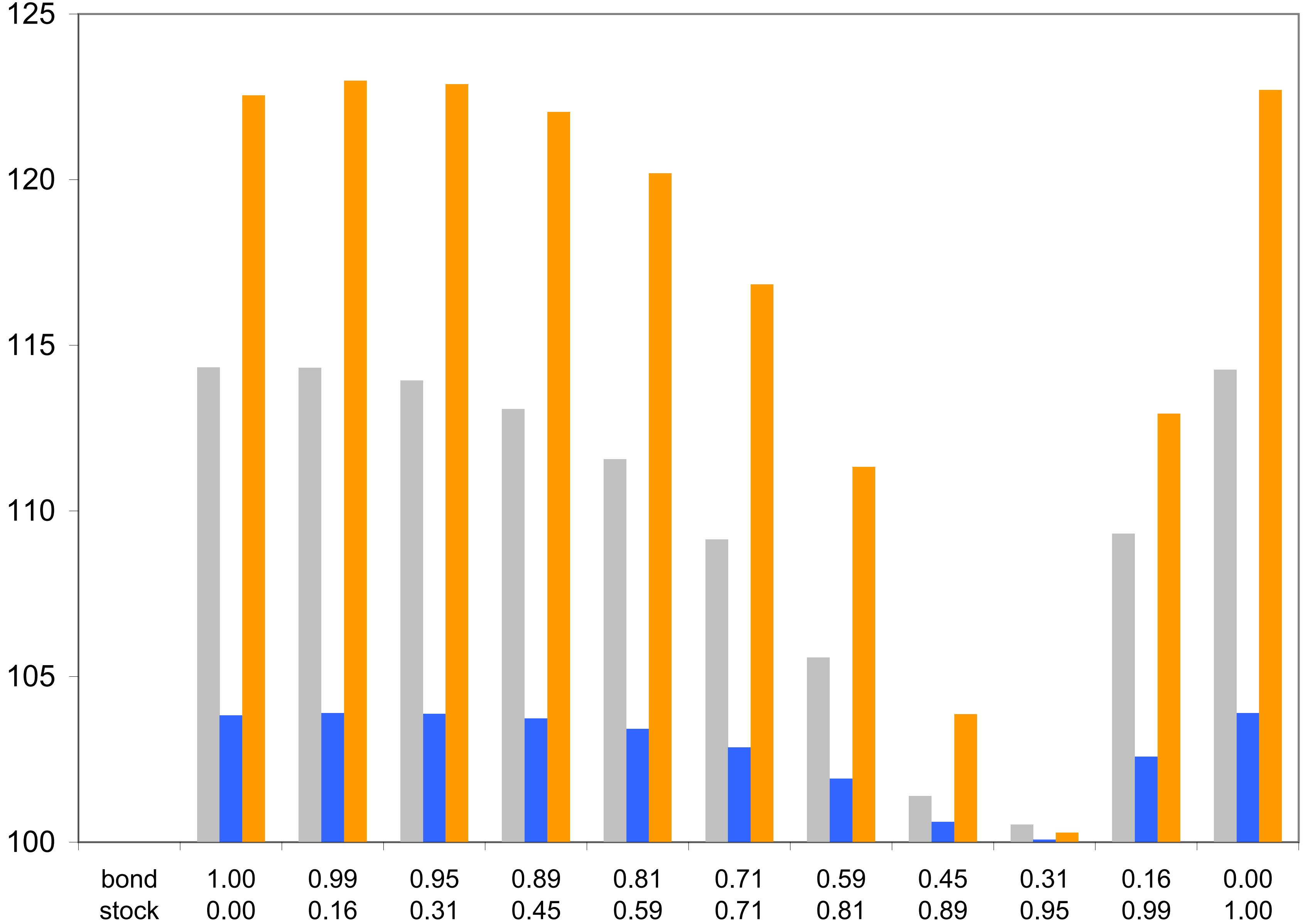
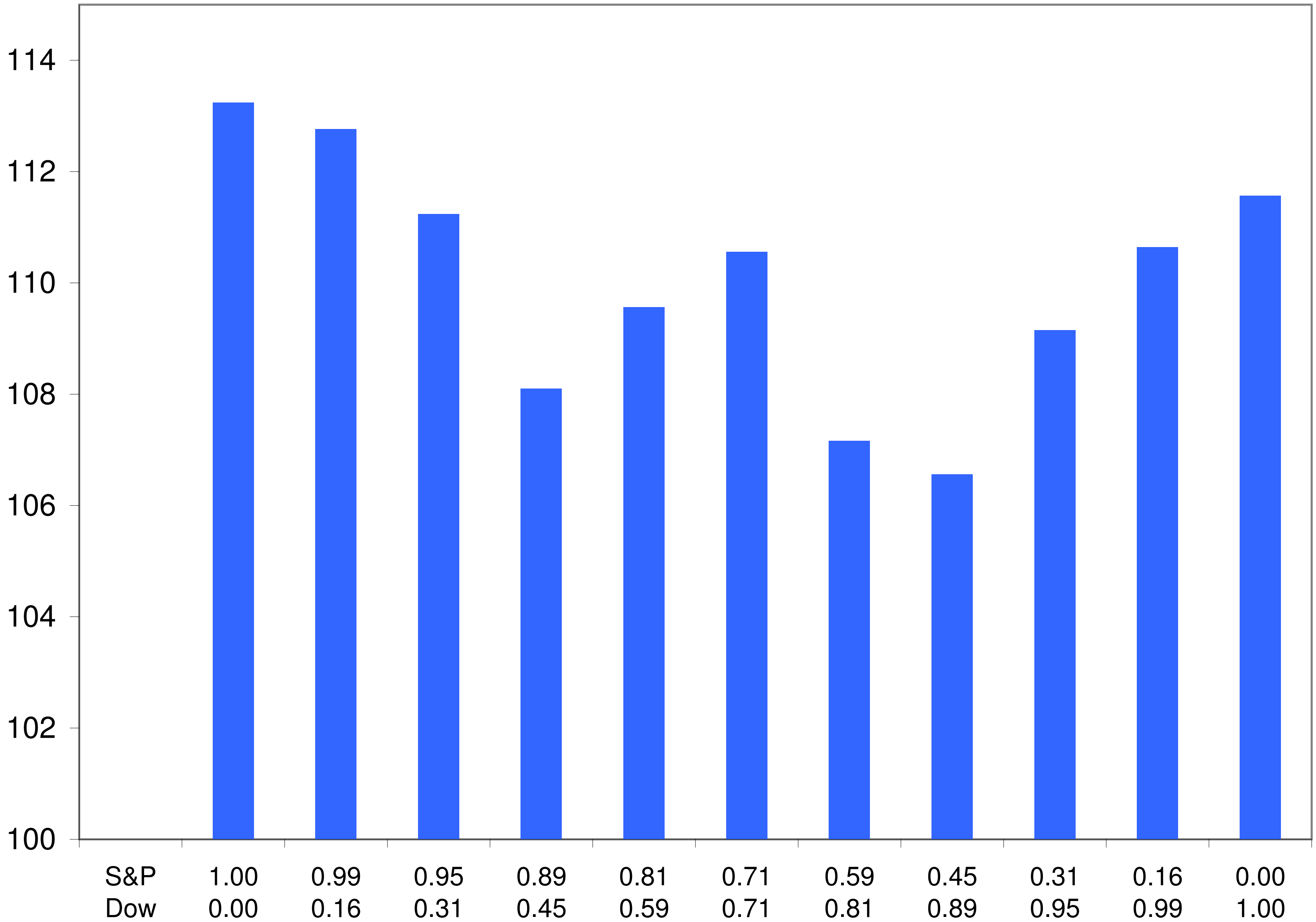


S&P500 and Dow Jones

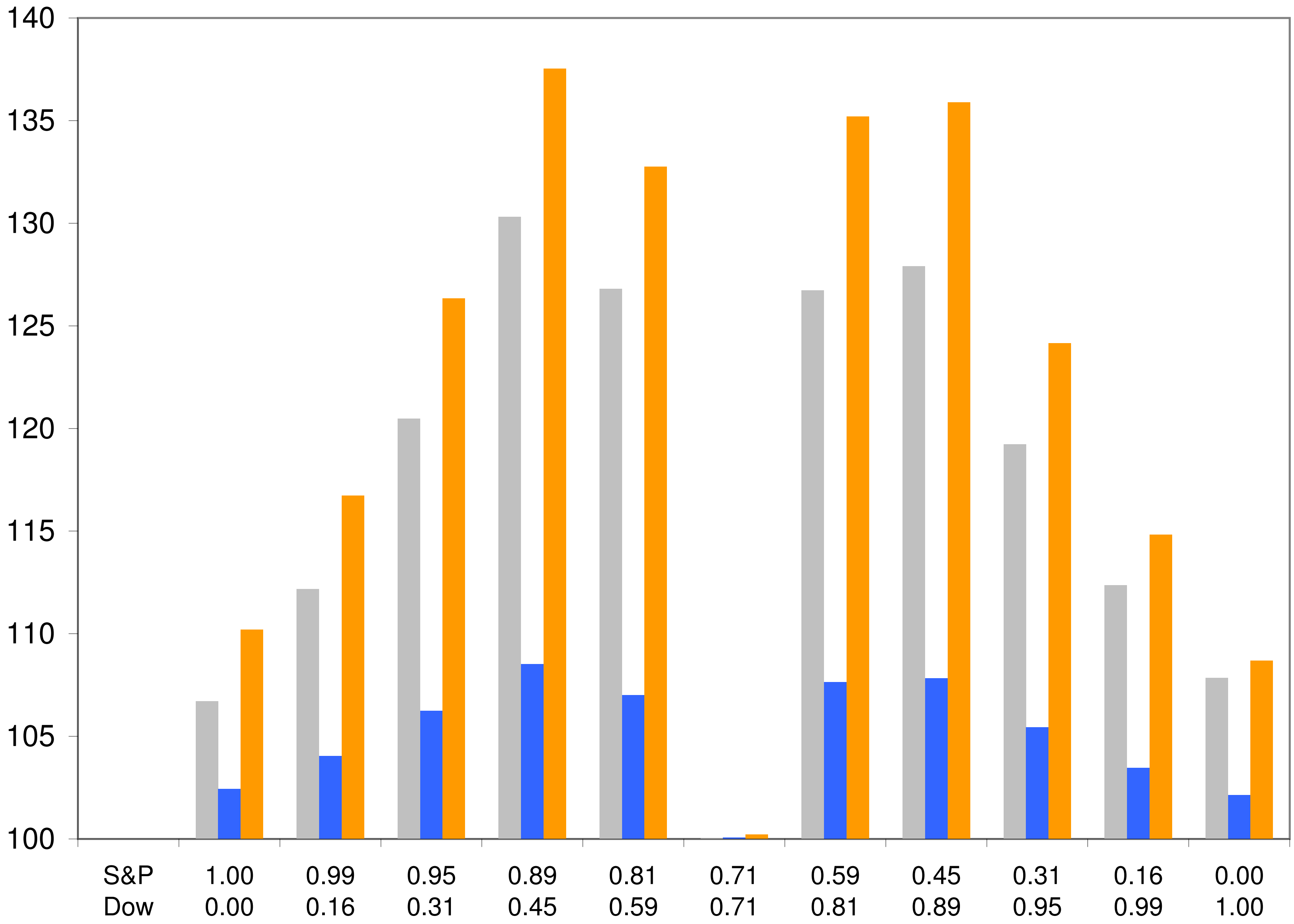
Correlation and return structure of equity indices is very different:

- Unconditional correlations are about 0.9.
- Asymmetry is greater.
- Expected returns are probably nearly equal.

SP & Dow (Full covariance)



SP & Dow (Correlations only)



More advanced questions

Would the value of correlations information be greater in more complex problems?

- Short sale constraints will reduce the value.
- No riskless asset can have either effect.
- Multi-period objective function should increase the value of correlations.

Conclusions

- The value of accurate daily correlations is moderate (maybe 20bp). Possibly why asset allocation is done monthly and ignores covariances.
- On some days, the value is much greater.
- Additional value may flow from multi-period optimization. See Colacito and Engle (2005) in progress.